Math 5329, Final (a)

Name Key

1. If $f(x) = (x-r)^m$, where $m > 1$, show that Newton's method will converge to the multiple root r, no matter where we start, but only linearly.

$$x_{n+1} = x_n - \frac{(x_n-r)^m}{m(x_n-r)^{m-1}} = x_n - \frac{1}{m}(x_n - r)$$

5

$$x_{n+1} - r = (x_n - r)\left(1 - \frac{1}{m}\right) \qquad e_{n+1} = \left(1-\frac{1}{m}\right)e_n$$

2. Give an example of a linear system $Ax = b$ for which the Jacobi iteration converges, although the matrix A is not diagonal dominant. (Hint: If $A = L + D + U$, the Jacobi method converges if and only if all eigenvalues of $D^{-1}(L+U)$ are less than one in absolute value.)

5

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \qquad -D^{-1}(L+U) = \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix} = -U$$

not diagonl dominant eigenvalues = (0, 0) so converges

3. Find the weights w_1, w_2 and sample points r_1, r_2 for the Gauss 2 point quadrature rule $\int_0^h f(x)\ dx \approx w_1 h f(r_1 h) + w_2 h f(r_2 h)$. (Hint: You should make some simplifing assumptions first, based on symmetry)

$W_1 = W_2 = \frac{1}{2}$

$r_1 = \frac{1}{2} - \sqrt{\frac{1}{12}}$

$r_2 = \frac{1}{2} + \sqrt{\frac{1}{12}}$

$\int_0^h f(x)\, dx = wh f(rh) + wh f((1-r)h)$

$h = \int_0^h 1\, dx = wh + wh$ $\quad 2w = 1$

5

$\frac{1}{2}h^2 = \int_0^h x\, dx = whrh + wh(1-r)h = wh^2$ $\quad w = \frac{1}{2}$

$\frac{1}{3}h^3 = \int_0^h x^2 dx = wh(rh)^2 + wh(1-r)^2h^2 = [wr^2 + w(1-r)^2]h^3$

$r^2 - r + \frac{1}{6} = 0$ $\quad r = \frac{1}{2} - \sqrt{\frac{1}{12}}$

4. Consider the multi-step method:
$U(t_{k+1}) = -4U(t_k) + 5U(t_{k-1}) + 4hf(t_k, U(t_k)) + 2hf(t_{k-1}, U(t_{k-1}))$

a. Calculate the truncation error. Is the method consistent?

$$T = \frac{u(t+h) + 4u(t) - 5u(t-h)}{6h} - \frac{2}{3}u'(t) - \frac{1}{3}u'(t-h)$$

5

$$= \frac{h^3 u^{iv}}{36}$$

yes, consistent

b. Determine if the method is stable.

4

$\lambda^2 + 4\lambda - 5 = 0$ $\quad \lambda = -5, 1$ so unstable

$(\lambda + 5)(\lambda - 1) = 0$

c. Is the method implicit or explicit?

1

explicit

5. If $L_N(x)$ is the Lagrange polynomial of degree N which interpolates to $f(x) = e^{2x}$ at $x = 1, 2, 3, \ldots N+1$, prove that $L_N(0)$ does NOT converge to $f(0) = 1$, as $N \to \infty$.

5

$$\left| L_N(x) - f(x) \right| = \left| \frac{(x-1)(x-2)\ldots(x-(N+1))}{(N+1)!} f^{(N+1)}(\xi) \right|$$

$$\left| L_N(0) - f(0) \right| = \left| \frac{(N+1)!}{(N+1)!} 2^{N+1} e^{2(\xi)} \right| \geq 2^{N+1}$$

$$0 \leq \xi \leq N+1$$

6. A 3 by 3 matrix A has eigenvalues very near 2, 8 and 9. If the shifted power method (note: not INVERSE shifted power method) $x_{n+1} = (A - pI)x_n$ is used, what value of p should be used if we want to maximize the rate of convergence to find the eigenvalue near 9? (Hint: the rate of convergence of the power method is the ratio of the second largest (in absolute value) eigenvalue to the largest.)

5

eigenvalues:

$-p+2$

$-p+8$

$-p+9$

$p = 5$ then eigenvalues of $A - pI$ are

-3

3

4

rate $= \frac{3}{4}$

7. The following MATLAB program solves a linear system $Ax = b$ using the Gauss-Jordan algorithm, in which A is reduced to diagonal form rather than upper triangular form, during the forward elimination (no pivoting in done in this program). For large N, approximately how many multiplications are done by this program? How does this algorithm compare in speed to normal Gauss elimination?

5

```
function x = GJ(A,b,N)
for i=1:N
   for j=1:N
      if (j==i)
         continue
      end
      r = A(j,i)/A(i,i);
      for k=i:N
         A(j,k) = A(j,k) - r*A(i,k);
      end
      b(j) = b(j) - r*b(i);
   end
end
for i=1:N
   x(i) = b(i)/A(i,i);
end
```

$$\sum_{i=1}^{N} \sum_{j=1}^{N} \sum_{k=i}^{N} 1 = \sum_{i=1}^{N} N(N-i) = N \tfrac{1}{2} N^2 = \tfrac{1}{2} N^3$$

$GE \sim \frac{1}{3} N^3$ so slower

Math 5329, Final (B)

Name Key

1\. a. Let $T_3(x)$ be the Taylor polynomial of degree 3 which matches $f(x), f'(x), f''(x)$ and $f'''(x)$ at $x = 0$, where $f(x) = sin(0.1\ x)$. Find the best possible bound on

2

$$max_{-0.5\le x\le 0.5}|T_3(x) - f(x)| \le \left|\frac{f^{IV}(\xi)}{4!}x^4\right| \le \frac{(0.1)^4 \sin(0.1\xi)}{24}(0.5)^4$$

$-0.5 \le \xi \le 0.5$, take $\xi = 0.5$, $x = 0.5$

$$\le 1.3\cdot 10^{-8}$$

b. Let $L_3(x)$ be the Lagrange polynomial of degree 3 which matches $f(x)$ at $x = -2, -0.5, 0.5$ and 2, where $f(x) = sin(0.1\ x)$. Find the best possible bound on

2

$$max_{-0.5\le x\le 0.5}|L_3(x) - f(x)| \le \left|\frac{f^{IV}(\xi)}{4!}(x+2)(x+\tfrac{1}{2})(x-\tfrac{1}{2})(x-2)\right| \le$$

$-2 \le \xi \le 2$ take $\xi = 2$, take $x = 0$

$$\frac{(0.1)^4 \sin(0.1\xi)}{24}(2)(\tfrac{1}{2})(\tfrac{1}{2})(2) \le 8.3\cdot 10^{-7}$$

2\. a. A root finder gives consecutive errors of $e_8 = 10^{-3}, e_9 = 10^{-5}, e_{10} = 10^{-11}$. Estimate the order of the method.

1

$10^{-5} = M(10^{-3})^\alpha$

$10^{-11} = M(10^{-5})^\alpha$ $\quad 10^6 = (10^2)^\alpha$

$\alpha = 3$

b. A quadrature method gives an error of 10^{-5} when $h = 10^{-2}$ and 10^{-11} when $h = 10^{-4}$. Estimate the (global) order of the method.

1

$\alpha = 3$ $\quad 10^6 = (10^2)^\alpha$

$10^{-5} = M(10^{-2})^\alpha$

$10^{-11} = M(10^{-4})^\alpha$

c. A differential equation solver gives an answer $u(1) = 0.148888$ when $h = 0.1$, and $u(1) = 0.140666$ when $h = 0.01$, and $u(1) = 0.140600$ when $h = 0.001$. Estimate the (global) order of the method.

1

$.008222 = M(0.1^\alpha - 0.01^\alpha)$ $\quad 0.148888 - I = M(0.1)^\alpha$

$.000066 = M(0.01^\alpha - 0.001^\alpha)$ $\quad 0.140666 - I = M(0.01)^\alpha$

$0.140600 - I = M(0.001)^\alpha$

$124.5 = 10^\alpha$

$\alpha = 2.1$

3. Consider the system

$$f_1(x_1, ..., x_N) = a_{11}x_1 + ... + a_{1N}x_N - b_1 = 0$$

$\cdot$

$\cdot$

$$f_N(x_1, ..., x_N) = a_{N1}x_1 + ... + a_{NN}x_N - b_N = 0$$

a. Write out Newton's method for solving this system, and show that it will converge in a single iteration (regardless of starting solution) to the solution of $Ax = b$, that is, to $A^{-1}b$.

2

$$x^{n+1} = x^n - A^{-1}(Ax^n - b) = A^{-1}b \quad (\text{exact for any } x^n)$$

b. Modify Newton's method by setting all off-diagonal terms of the Jacobian matrix equal to zero, and show that it is now equivalent to the Jacobi iteration for solving $Ax = b$.

2

$$x_i^{n+1} = x_i^n - \frac{1}{a_{ii}}\left(\sum_{j=1}^{N} a_{ij} x_j^n - b_i\right)$$

$$x_i^{n+1} = \frac{-\sum_{j \neq i} a_{ij} x_j^n + b_i}{a_{ii}} \quad \leftarrow \text{Jacobi method}$$

4. a. Is the following method stable? (Justify answer)

$$\frac{U_{k+1}+4U_k-5U_{k-1}}{6h} = \frac{2}{3}f(t_k, U_k) + \frac{1}{3}f(t_{k-1}, U_{k-1})$$

2 $\lambda^2 + 4\lambda - 5 = 0$ $\lambda = -5, 1$ no

b. Is the method consistent (with $u' = f(t,u)$)? (Justify answer)

2 $T = \frac{h^3 u^{iv}}{36}$ yes

c. Will the finite difference solution converge to the differential equation solution as $h \to 0$?

1 no

5. a. Will the iteration $x_{n+1} = 2 + (x_n - 2)^4$ converge when x_0 is sufficiently close to the root $r = 2$? If so, what is the order of convergence? (Justify your answer theoretically)

2 $g(x) = 2 + (x-2)^4$ $g''(x) = 12(x-2)^2$ $g^{iv}(x) = 24$

$g'(x) = 4(x-2)^3$ $g'''(x) = 24(x-2)$ yes order = 4

b. What is the range of values of x_0 for which this iteration will converge to $r = 2$?

1 $1 \le x_0 < 3$ $e_{n+1} = e_n^4$ converge if $|e_0| < 1$

c. There is a second root, that is, another value such that if $x_n = r$, $x_{n+1} = r$ also. Find this root, and the range of values of x_0 for which the iteration will converge to this root.

1 $r = 3$ $x_0 = 3$ only (actually for $x_0 = 1$ also)

Math 5329, Final (c)

Name Key

1. a. Let $T_3(x)$ be the Taylor polynomial of degree 3 which matches $f(x), f'(x), f''(x)$ and $f'''(x)$ at $x = 0$, where $f(x) = cos(4x)$. Find the best possible bound on

2

$max_{-0.5\leq x\leq 0.5}|T_3(x) - f(x)| \leq \left|\frac{f^{iv}(\xi)}{4!} x^4\right| \leq \frac{4^4 \cos(4\xi)}{24}\left(\frac{1}{2}\right)^4 \leq \frac{2}{3}$

b. Let $L_3(x)$ be the Lagrange polynomial of degree 3 which matches $f(x)$ at $x = -2, -0.5, 0.5$ and 2, where $f(x) = cos(4x)$. Find the best possible bound on

2

$max_{-0.5\leq x\leq 0.5}|L_3(x) - f(x)| \leq \left|\frac{f^{iv}(\xi)}{4!}(x+2)(x+\tfrac{1}{2})(x-\tfrac{1}{2})(x-2)\right|$

$\leq \frac{4^4 \cos(4\xi)}{24} 2\left(\tfrac{1}{2}\right)\left(\tfrac{1}{2}\right)(2) \leq 10,666$

2. a. A root finder gives consecutive errors of $e_8 = 10^{-3}, e_9 = 10^{-5}, e_{10} = 10^{-11}$. Estimate the order of the method.

1

$10^{-5} = M(10^{-3})^\alpha$

$10^{-11} = M(10^{-5})^\alpha$

$10^6 = (10^2)^\alpha$ $\alpha = 3$

b. A quadrature method gives an error of 10^{-5} when $h = 10^{-2}$ and 10^{-11} when $h = 10^{-4}$. Estimate the order of the method.

1

$10^{-5} = M(10^{-2})^\alpha$

$10^{-11} = M(10^{-4})^\alpha$

$10^6 = (10^2)^\alpha$ $\alpha = 3$

3. Use the inverse power method to determine the smallest eigenvalue of

$$A = \begin{bmatrix} \frac{4}{6} & \frac{-2}{6} \\ \frac{-5}{6} & \frac{1}{6} \end{bmatrix}$$

$A^{-1} = \begin{pmatrix} -1 & -2 \\ -5 & -4 \end{pmatrix}$

Start the iteration with $(x_0; y_0) = (3, 8)$.

2

$\begin{pmatrix} 3 \\ 8 \end{pmatrix} \to \begin{pmatrix} -19 \\ -47 \end{pmatrix} \to \begin{pmatrix} 113 \\ 283 \end{pmatrix}$

ratio ≈ -6

$\lambda_{min} = -\frac{1}{6}$

4. Take one step of a third order Taylor series method (Euler is the first order Taylor method) to approximate the solution of the following problem, at $t = 1.01$:

$u' = 2tu$

$u(1) = 1$

$u' = 2tu = 2$

$u'' = 2tu' + 2u = 2(2) + 2(1) = 6$

$u''' = 2tu'' + 4u' = 2(6) + 4(2) = 20$

2

$u(1+h) = u(1) + u'(1)h + u''(1)\frac{h^2}{2} + u'''(1)\frac{h^3}{6}$

$= 1 + 2h + \frac{6}{2}h^2 + \frac{20}{6}h^3 = 1.02030333$

5. Consider the multi-step method:
$8U(t_{k+1}) = 9U(t_k) - U(t_{k-2}) + 3hf(t_{k+1}, U(t_{k+1}) + 6hf(t_k, U(t_k)) - 3hf(t_{k-1}, U(t_{k-1}))$

a. Is the method consistent? yes

3

$T = \frac{h^4 u^v}{30}$

b. Is the method is stable?

$8\lambda^3 - 9\lambda^2 + 1 = 0$

stable

$(\lambda-1)(8\lambda^2-\lambda-1) = 0$

$\lambda = 1$ or $\frac{1 \pm \sqrt{33}}{16} = 0.421$
-0.296

c. Is the method implicit or explicit?

Implicit

6. Do one iteration of Newton's method, starting from $(0, 0)$, to solve:
$f(x, y) = 2x^2 + y - 1 = 0$
$g(x, y) = -2x + y^2 + 1 = 0$

2

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} x_n \\ y_n \end{pmatrix} - \begin{pmatrix} 4x & 1 \\ -2 & 2y \end{pmatrix}_n^{-1} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}^{-1} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}$$

7. Write $(1 + x)^{1/3} - 1$ in a form where there is no serious problem with roundoff, when $x \approx 0$. (Hint: $a^3 - b^3 = (a^2 + ab + b^2)(a - b)$)

2

$$(1+x)^{\frac{1}{3}} - 1 = \frac{(1+x) - 1^3}{(1+x)^{\frac{2}{3}} + (1+x)^{\frac{1}{3}} + 1} = \frac{x}{(1+x)^{\frac{2}{3}} + (1+x)^{\frac{1}{3}} + 1}$$

8. Determine the (global) order of the quadrature rule:

$\int_a^b f(x)dx \approx \sum_{i=1}^{N} \left[\frac{h}{4} f(x_{i-1}) + \frac{3h}{4} f(x_{i-1} + \frac{2h}{3})\right]$

$\int_0^h f(x)\,dx = \frac{h}{4} f(0) + \frac{3h}{4} f\left(\frac{2h}{3}\right)$

$f=1$ $\quad h \stackrel{?}{=} \frac{h}{4} + \frac{3h}{4} = h$ ✓

$f=x$ $\quad \frac{1}{2}h^2 \stackrel{?}{=} \frac{3h}{4}\frac{2h}{3} = \frac{1}{2}h^2$ ✓

2

$f=x^2$ $\quad \frac{1}{3}h^3 \stackrel{?}{=} \frac{3h}{4}\left(\frac{2h}{3}\right)^2 = \frac{1}{3}h^3$

$f=x^3$ $\quad \frac{1}{4}h^4 = \frac{3h}{4}\left(\frac{2h}{3}\right)^3 = \frac{2}{9}h^4$

$O(h^3)$

9. Will the iteration $x_{n+1} = 2 - \frac{3}{2}x_n + \frac{1}{2}x_n^3$ converge to the root 1, if the starting guess is sufficiently good? **Justify** your answer.

$g(x) = 2 - \frac{3}{2}x + \frac{1}{2}x^3$

$g'(x) = -\frac{3}{2} + \frac{3}{2}x^2$

2

$g'(1) = 0$ so yes

10. Will the following iteration converge (to something)? Justify your answer *theoretically.*

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 0.5 & 0.75 \\ -0.75 & 0.5 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix} + \begin{bmatrix} 2 \\ -4 \end{bmatrix}$$

2

$(0.5-\lambda)(0.5-\lambda) + 0.75^2 = 0$

$\lambda = 0.5 \pm 0.75 i$ $\qquad |\lambda| = 0.901$

Converges

Math 5329 Final (2)

Name Key

1. If $f(x) = (x-r)^m$, where $m > 1$, show that Newton's method will converge to the multiple root r, no matter where we start, but only linearly.

$$x_{n+1} = x_n - \frac{(x_n - r)^m}{m(x_n-r)^{m-1}} = x_n - \frac{1}{m}(x_n - r)$$

2

$$e_{n+1} = e_n - \frac{1}{m}e_n = \left(1-\frac{1}{m}\right)e_n$$

$$0 < 1-\frac{1}{m} < 1$$ so converges linearly

2. Find the optimum weights w_1, w_2 and sample points r_1, r_2 for the (Gauss) 2 point quadrature rule $\int_0^h f(x)\,dx \approx w_1 h f(r_1 h) + w_2 h f(r_2 h)$. (Hint: You can make some simplifing assumptions first, based on symmetry)

2

$w_1 = w_2 = w$
$r_1 = r$
$r_2 = 1-r$

$$\int_0^h f(x)\,dx \approx wh\,f(rh) + wh\,f((1-r)h)$$

$f=1$: $h = wh + wh \Rightarrow w = \frac{1}{2}$

$f=x$: $\frac{1}{2}h^2 = wh(rh) + wh(1-r)h = wh^2 \Rightarrow w = \frac{1}{2}$

$f=x^2$: $\frac{1}{3}h^3 = whr^2h^2 + wh(1-2r+r^2)h^2 \Rightarrow \frac{1}{3} = \frac{1}{2}r^2 + \frac{1}{2}(1-2r+r^2)$

$\frac{1}{3} = r^2 - r + \frac{1}{2}$

$r^2 - r + \frac{1}{6} = 0$ $\quad r = \frac{1}{2} - \sqrt{\frac{1}{12}}$

3. a. A root finder gives consecutive errors of $e_6 = 10^{-2}, e_7 = 10^{-5}, e_8 = 10^{-13}$. Estimate the order of the method.

1

$10^{-5} = M(10^{-2})^\alpha$
$10^{-13} = M(10^{-5})^\alpha$

$10^8 = (10^3)^\alpha$ $\quad \alpha = \frac{8}{3}$

b. A quadrature method gives an error of 10^{-4} when $h = 10^{-2}$ and 10^{-11} when $h = 10^{-4}$. Estimate the (global) order of the method.

1

$10^{-4} = M(10^{-2})^\alpha$
$10^{-11} = M(10^{-4})^\alpha$

$10^7 = (10^2)^\alpha$ $\quad \alpha = \frac{7}{2}$

4. a. Is the following method stable? (Justify answer)

$$\frac{U_{k+1}-U_k}{h} = f(t_k, U_k) + f(t_{k-1}, U_{k-1})$$

$\lambda - 1 = 0 \quad \lambda = 1$ yes

1

b. Is the method consistent (with $u' = f(t,u)$)? (Justify answer)

no l.h.s $\approx u'$, r.h.s $\approx 2f(t,u)$

1

c. Will the finite difference solution converge to the differential equation solution as $h \to 0$? no

1

5. Let $T_3(x)$ be the Taylor polynomial of degree 3 which matches $f(x), f'(x), f''(x)$ and $f'''(x)$ at $x = 0$, where $f(x) = x^4$. Find the best possible bound on

$max_{-0.5 \le x \le 0.5} |T_3(x) - f(x)| \le$ $\left|\frac{f^{(4)}(\xi)}{4!} x^4\right| \le \left|\frac{24}{24} x^4\right| \le \left(\frac{1}{2}\right)^4 = \frac{1}{16}$

2

6. Use the inverse power method to determine the smallest eigenvalue of

2

$$A = \begin{bmatrix} 4 & 3 \\ 3 & 2 \end{bmatrix} \qquad A^{-1} = \frac{\begin{bmatrix} 2 & -3 \\ -3 & 4 \end{bmatrix}}{-1} = \begin{pmatrix} -2 & 3 \\ 3 & -4 \end{pmatrix}$$

Start the iteration with $(x_0, y_0) = (1, 1)$.

$$x_1 = \begin{pmatrix} -2 & 3 \\ 3 & -4 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad x_2 = \begin{pmatrix} -5 \\ 7 \end{pmatrix} \qquad x_3 = \begin{pmatrix} 31 \\ -43 \end{pmatrix}$$

$$\frac{31}{-5} = -6.2 \qquad \frac{-43}{7} = -6.14 \qquad \lambda \simeq \frac{1}{-6.2} \simeq -0.16$$

7. Give an example of a linear system $Ax = b$ for which the Jacobi iteration converges, although the matrix A is not diagonal dominant. (Hint: If $A = L + D + U$, the Jacobi method converges if and only if all eigenvalues of $D^{-1}(L + U)$ are less than one in absolute value.)

2

$$\begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix}$$

$$D^{-1}(L+U) = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \qquad \lambda = 0, 0$$

so converges

8. Do one iteration of Newton's method, starting from $(0,0)$, to solve:

$f(x,y) = 2x^3 + y - 1 = 0$
$g(x,y) = -2x + y^3 + 1 = 0$

2

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} - \begin{pmatrix} 6x^2 & 1 \\ -2 & 3y^2 \end{pmatrix}^{-1}_{(0,0)} \begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -0.5 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 1 \end{pmatrix}$$

9. If $p_N(x)$ is the polynomial of degree N which interpolates to $f(x) = e^{\alpha x}$ at $x = 1, 2, 3, ..., N+1$, use the Lagrange error formula to show that $p_N(0)$ does NOT converge to $f(0)$ if $\alpha > 1$.

2

$$|p_N(0) - f(0)| = \left| \frac{f^{N+1}(\psi)}{(N+1)!} (0-1)(0-2)\dots(0-(N+1)) \right|$$

$$= \frac{\alpha^{N+1} e^{\alpha\psi}}{(N+1)!} (N+1)! = \alpha^{N+1} e^{\alpha\psi} \geq \alpha^{N+1} \to \infty$$

$$(0 < \psi < N+1)$$

10. Write $[\sqrt{1+x} - 1]/x$ in a form such that there is not a serious problem with roundoff error when x is close to 0.

1

$$\frac{\sqrt{1+x} - 1}{x} \frac{(\sqrt{1+x} + 1)}{\sqrt{1+x} + 1} = \frac{1}{\sqrt{1+x} + 1}$$