Math 5329, Final

Name ______________________________

1. a. Let $T_3(x)$ be the Taylor polynomial of degree 3 which matches $f(x), f'(x), f''(x)$ and $f'''(x)$ at $x = 0$, where $f(x) = cos(4x)$. Find the best possible bound on

   $max_{-0.5 \leq x \leq 0.5} |T_3(x) - f(x)| \leq$

   b. Let $L_3(x)$ be the Lagrange polynomial of degree 3 which matches $f(x)$ at $x = -2, -0.5, 0.5$ and 2, where $f(x) = cos(4x)$. Find the best possible bound on

   $max_{-0.5 \leq x \leq 0.5} |L_3(x) - f(x)| \leq$

2. a. A root finder gives consecutive errors of $e_8 = 10^{-3}, e_9 = 10^{-5}, e_{10} = 10^{-11}$. Estimate the order of the method.

   b. A quadrature method gives an error of 10^{-5} when $h = 10^{-2}$ and 10^{-11} when $h = 10^{-4}$. Estimate the order of the method.

3. Use the inverse power method to determine the smallest eigenvalue of

$$A = \begin{bmatrix} \frac{4}{6} & \frac{-2}{6} \\ \\ \frac{-5}{6} & \frac{1}{6} \end{bmatrix}$$

Start the iteration with $(x_0, y_0) = (3, 8)$.

4. Take one step of a third order Taylor series method (Euler is the first order Taylor method) to approximate the solution of the following problem, at $t = 1.01$:

$u' = 2tu$
$u(1) = 1$

5. Consider the multi-step method:
$8U(t_{k+1}) = 9U(t_k) - U(t_{k-2}) + 3hf(t_{k+1}, U(t_{k+1}) + 6hf(t_k, U(t_k)) - 3hf(t_{k-1}, U(t_{k-1}))$

a. Is the method consistent?

b. Is the method is stable?

c. Is the method implicit or explicit?

6. Do one iteration of Newton's method, starting from $(0, 0)$, to solve:
   $f(x, y) = 2x^2 + y - 1 = 0$
   $g(x, y) = -2x + y^2 + 1 = 0$

7. Write $(1 + x)^{1/3} - 1$ in a form where there is no serious problem with roundoff, when $x \approx 0$. (Hint: $a^3 - b^3 = (a^2 + ab + b^2)(a - b)$)

8. Determine the (global) order of the quadrature rule:
$\int_a^b f(x)dx \approx \sum_{i=1}^N \left[\frac{h}{4}f(x_{i-1}) + \frac{3h}{4}f(x_{i-1}+\frac{2h}{3})\right]$

9. Will the iteration $x_{n+1} = 2 - \frac{3}{2}x_n + \frac{1}{2}x_n^3$ converge to the root 1, if the starting guess is sufficiently good? **Justify** your answer.

10. Will the following iteration converge (to something)? Justify your answer *theoretically.*

$$\begin{bmatrix} x_{n+1} \\ y_{n+1} \end{bmatrix} = \begin{bmatrix} 0.5 & 0.75 \\ -0.75 & 0.5 \end{bmatrix} \begin{bmatrix} x_n \\ y_n \end{bmatrix} + \begin{bmatrix} 2 \\ -4 \end{bmatrix}$$