Math 4329, Test II (e)

Name Key

1. If $P_2(x)$ is the second degree polynomial that interpolates to $f(x) = \frac{6}{1+x}$ at $x = 0, 0.1, 0.2$, find a reasonable bound on the error at $x = 0.15$.

$f''' = \frac{-36}{(1+x)^4}$

$$|f(x) - P_2(x)| = \left|\frac{f'''(\xi)}{3!}(x-0)(x-0.1)(x-0.2)\right|$$

$0 \le \xi \le 0.2$

$$\le \frac{36}{6}(0.15)(0.05)(0.05) = 0.00225$$

2. Find A, B, C such that the approximation $u'(t) \approx \frac{Au(t)+Bu(t-h)+Cu(t-2h)}{h}$ is as high order as possible.

$A \quad u = u$

$B \quad u(t-h) = u - u'h + u''\frac{h^2}{2} + \ldots$

$C \quad u(t-2h) = u - 2u'h + 4u''\frac{h^2}{2} + \ldots$

$$\frac{(A+B+C)u + (-B-2C)u'h + (B+4C)u''\frac{h^2}{2} + \ldots}{h}$$

$$\begin{cases} A+B+C=0 \\ -B-2C=1 \\ B+4C=0 \end{cases}$$

$C = \frac{1}{2}$

$B = -2$

$A = \frac{3}{2}$

3. Find A, r which make the approximation

$\int_{-1}^{1} f(x)dx \approx Af(-r) + Af(r)$

as high degree of precision as possible (thus as high order as possible).

$2 = \int_{-1}^{1} 1\,dx = A + A$

$0 = \int_{-1}^{1} x\,dx = A(-r) + A(r) = 0$

$\frac{2}{3} = \int_{-1}^{1} x^2\,dx = A(-r)^2 + Ar^2 = 2r^2$

$A = 1$

$r = \sqrt{\frac{1}{3}}$

4. True or False:

F a. The experimental order of convergence is $O(h^3)$ if a quadrature rule yields errors of 0.0032 when $h = 0.1$ and 0.0002 when $h = 0.05$.

F b. The Gauss-Seidel iterative method (for $Ax = b$) is generally slower than the Jacobi method.

F c. The Jacobi iterative method (for $Ax = b$) converges only if the matrix is diagonal-dominant.

T d. Roundoff error is much more serious, in general, for derivative approximations than for integral approximations.

T e. Gaussian elimination, when applied to a general N by N linear system, requires $O(N^3)$ arithmetic operations.

F f. If $s(x)$ is a cubic spline, then s, s', s'' and s''' must be continuous everywhere.

F g. If a quadrature method is exact for all polynomials of degree n, its global error is $O(h^n)$ for general smooth functions.

F h. If a matrix A has condition number 10, we expect to lose about 10 significant digits in solving $Ax = b$ with Gauss elimination and partial pivoting.

5. a. Write out the Jacobi iteration, for the system

$$\begin{bmatrix} 4 & 3 & -1 \\ 1 & 4 & -1 \\ -1 & 2 & 7 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ 9 \\ -8 \end{bmatrix}$$

Will it converge? Explain.

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \\ z_{n+1} \end{pmatrix} = \underbrace{\begin{pmatrix} 0 & -\frac{3}{4} & \frac{1}{4} \\ -\frac{1}{4} & 0 & \frac{1}{4} \\ \frac{1}{7} & -\frac{2}{7} & 0 \end{pmatrix}}_{B} \begin{pmatrix} x_n \\ y_n \\ z_n \end{pmatrix} + \begin{pmatrix} 7/4 \\ 9/4 \\ -8/7 \end{pmatrix}$$

$\lambda(B) = 0.06$

0.36

-0.42

so $|\lambda_{max}(B)| < 1$

$\Rightarrow$ converges

b. Write out the Gauss-Seidel iteration, for this system

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \\ z_{n+1} \end{pmatrix} \begin{matrix} = -\frac{3}{4}x_n + \frac{1}{4}y_n + \frac{7}{4} \\ = -\frac{1}{4}x_{n+1} + \frac{1}{4}y_n + \frac{9}{4} \\ = \frac{1}{7}x_{n+1} - \frac{2}{7}y_{n+1} - \frac{8}{7} \end{matrix}$$

Math 4329, Test II (f)

Name Key

1. a. Suppose $P_3(x)$ is the polynomial of degree 3 which interpolates $f(x) = cos(3x)$ at $x_0 = -1.5h, x_1 = -0.5h, x_2 = 0.5h, x_3 = 1.5h$. Find a reasonable upper bound on the error for $x_0 < x < x_3$. You can be "lazy," that is, bound $|x - x_i|$ for each i by $|x_3 - x_0|$.

5

$$|f(x)-P_3(x)| \le \left|\frac{f^{IV}(\xi)}{4!}(x-x_0)(x-x_1)(x-x_2)(x-x_3)\right| \le \left|\frac{3^4 \cos(3\xi)}{24}(3h)^4\right|$$

$$\le \frac{3^8}{24}h^4 = 273.375\, h^4$$

b. [Extra credit] Same question, but now, DON'T be lazy, get the best possible bound on $|q(x)| \equiv |(x-x_0)(x-x_1)(x-x_2)(x-x_3)|$.

3

$$g(x) = (x+\tfrac{3h}{2})(x+\tfrac{h}{2})(x-\tfrac{h}{2})(x-\tfrac{3h}{2}) = (x^2-\tfrac{9h^2}{4})(x^2-\tfrac{h^2}{4})$$

$$g'(x) = 2x\left[2x^2 - \tfrac{10h^2}{4}\right] = 0 \Rightarrow x=0 \text{ or } x = \pm\sqrt{\tfrac{5}{4}}\,h$$

$$g(0) = \frac{9h^4}{16} \quad g(\pm\sqrt{\tfrac{5}{4}}h) = -h^4 \quad \text{so}$$

$$|f(x)-P_3(x)| \le \frac{3^4}{24}h^4 = 3.375\, h^4$$

2. Find A, B which make the approximation.

$\int_0^h f(x)dx \approx Ahf(0) + Bhf(\frac{2h}{3})$

5

as high order as possible. With your choice of A, B, what is the degree of precision, and what is the order of the error (power of h that the global error is proportional to) in this approximation?

$f=1$ $\quad h = \int_0^h 1\,dx = Ah + Bh$ $\quad A+B=1$

$f=x$ $\quad \frac{1}{2}h^2 = \int_0^h x\,dx = Bh\,\frac{2h}{3}$ $\quad \frac{2}{3}B = \frac{1}{2}$

$f=x^2$ $\quad \frac{1}{3}h^3 = \int_0^h x^2 dx \doteq Bh\left(\frac{2h}{3}\right)^2$ $\quad \frac{4}{9}B = \frac{1}{3}$

$f=x^3$ $\quad \frac{1}{4}h^4 = \int_0^h x^3 dx = Bh\left(\frac{2h}{3}\right)^3$ $\quad \frac{8}{27}B \cancel{=} \frac{1}{4}$

$B = \frac{3}{4}$
$A = \frac{1}{4}$

degree of precision = 2
so error $O(h^3)$

3. Consider the linear system:

$$\begin{bmatrix} 1+\epsilon & 1 \\ 1 & 1+\epsilon \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

a. Write out the equations for the Jacobi iterative method for solving this system (don't actually do any iterations).

2

$$x_{n+1} = (3 - y_n)/(1+\epsilon)$$
$$y_{n+1} = (4 - x_n)/(1+\epsilon)$$

b. Write out the equations for the Gauss-Seidel iterative method for solving this system.

1

$$x_{n+1} = (3 - y_n)/(1+\epsilon)$$
$$y_{n+1} = (4 - x_{n+1})/(1+\epsilon)$$

c. True or False: If $\epsilon > 0$, the Jacobi iterative method (3a) will converge for *any* starting vector (x_0, y_0). Give a reason for your answer.

1

true, A is diagonal dominant

d. Find the condition number of the above matrix (using the L_∞ norm). If you were to solve the above linear system using Gaussian elimination with partial pivoting, would you expect serious roundoff errors, if ϵ is very small?

2

Hint: The inverse of

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \qquad \|A\| = 2+\epsilon$$

is

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$A^{-1} = \frac{1}{2\epsilon+\epsilon^2}\begin{bmatrix} 1+\epsilon & -1 \\ -1 & 1+\epsilon \end{bmatrix} \qquad \|A^{-1}\| = \frac{2+\epsilon}{2\epsilon+\epsilon^2} = \frac{1}{\epsilon}$$

2

$$\frac{2+\epsilon}{\epsilon} = \text{cond}(A)$$

yes, serious roundoff

4. The following function is a cubic spline for what values of a, b, c?

4

$$
\begin{aligned}
s(x) &= 2x^3 - 3x^2 + 3x - 6 \quad for \quad 0 < x \leq 1 \\
&= x^3 + ax^2 + bx + c \quad for \quad 1 < x \leq 2
\end{aligned}
$$

at $x=1$

$s = 2x^3 - 3x^2 + 3x - 6 \qquad -4 = 1 + a + b + c \qquad x^3 + ax^2 + bx + c$

$s' = 6x^2 - 6x + 3 \qquad 3 = 3 + 2a + b \qquad 3x^2 + 2ax + b$

$s'' = 12x - 6 \qquad 6 = 6 + 2a \qquad 6x + 2a$

$a = 0$
$b = 0$
$c = -5$

5. [Extra Credit] Use Taylor series expansions to determine the error in the approximation

$$u^{iv}(t) \approx \frac{u(t+2h) - 4u(t+h) + 6u(t) - 4u(t-h) + u(t-2h)}{h^4} = \frac{h^4 u^{iv} + \frac{1}{6} h^6 u^{vi} + \dots}{h^4}$$

(3)

Hint: expand $u(t+2h)$, etc, out to the h^6 term.

$= u^{iv} + \frac{1}{6} h^2 u^{vi} + \dots$

error

Math 4329, Test II (g) Name Key

Do problem 5 and any 3 of the first 4. Mark clearly which 3 to grade, no extra credit for doing all 4.

1. Use Taylor series expansions to determine the error in the approximation

$$u^{iv}(t) \approx \frac{u(t+2h)-4u(t+h)+6u(t)-4u(t-h)+u(t-2h)}{h^4}$$

Hint:

$u(t-2h) = u-2hu'+4h^2u''/2-8h^3u'''/6+16h^4u^{iv}/24-32h^5u^v/120+64h^6u^{vi}/720...$
$u(t-h) \quad = u-hu'+\ h^2u''/2-\ h^3u'''/6+\ h^4u^{iv}/24-\ h^5u^v/120+\ h^6u^{vi}/720...$
$u(t) \qquad = u$
$u(t+h) \quad = u+hu'+\ h^2u''/2+\ h^3u'''/6+\ h^4u^{iv}/24+\ h^5u^v/120+\ h^6u^{vi}/720...$
$u(t+2h) = u+2hu'+4h^2u''/2+8h^3u'''/6+16h^4u^{iv}/24+32h^5u^v/120+64h^6u^{vi}/720...$

$$rhs = \frac{0u+0hu'+0h^2u''+0h^3u'''+h^4u^{iv}+\frac{1}{6}h^6u^{vi}}{h^4} = u^{iv}+\frac{1}{6}h^2u^{vi}$$

so error $= \frac{1}{6}h^2u^{vi}+\ldots$

2. If $p_N(x)$ is the polynomial of degree N which interpolates $f(x) = cos(3x)$ at $N+1$ uniformly spaced points between 0 and π, find a bound, involving only N, on $max(0 \le x \le \pi)|p_N(x)-f(x)|$ Will your bound go to zero as $N \to \infty$?

$$\left|\frac{f^{n+1}(\xi)}{(n+1)!}(x-x_0)\ldots(x-x_n)\right| \le \frac{3^{n+1}}{(n+1)!}\pi^{n+1} = \frac{(3\pi)^{n+1}}{(n+1)!}$$

yes, $\to 0$ as $n \to \infty$

3. Determine the equations which must be satisfied for

$$s(x) = \begin{cases} a(x-2)^2 + b(x-1)^3 & x \le 1 \\ c(x-2)^2 & 1 \le x \le 3 \\ d(x-2)^2 + e(x-3)^3 & 3 \le x \end{cases}$$

to be a cubic spline. $\quad x=1 \qquad x=3$

$$s \qquad a(x-2)^2 + b(x-1)^3 \qquad c(x-2)^2 \qquad d(x-2)^2 + e(x-3)^3$$

$$s' \qquad 2a(x-2) + 3b(x-1)^2 \qquad 2c(x-2) \qquad 2d(x-2) + 3e(x-3)^2$$

$$s'' \qquad 2a + 6b(x-1) \qquad 2c \qquad 2d + 6e(x-3)$$

$$s \qquad a(-1)^2 = c(-1)^2 \qquad c(1)^2 = d(1)^2 \qquad a = c = d$$

$$s' \qquad 2a(-1) = 2c(-1) \qquad 2c(1) = 2d(1)$$

$$s'' \qquad 2a = 2c \qquad 2c = 2d$$

4. Find A, B, C which make the approximation

$$\int_0^h f(x)dx \approx Ahf(0) + Bhf(0.4h) + Chf(0.8h)$$

as high order as possible.

$$h = \int_0^h 1\,dx = Ah + Bh + Ch$$

$$\tfrac{1}{2}h^2 = \int_0^h x\,dx = Ah(0) + Bh(0.4h) + Ch(0.8h)$$

$$\tfrac{1}{3}h^3 = \int_0^h x^2 dx = Ah(0)^2 + Bh(0.16h^2) + Ch(0.64h^2)$$

$$A + B + C = 1$$

$$-2.5\left\{\begin{array}{l} 0.4B + 0.8C = 0.5 \\ 0.16B + 0.64C = \frac{1}{3} \end{array}\right.$$

$$C = \frac{5}{12} \qquad B = \frac{5}{12} \qquad A = \frac{2}{12}$$

5. True or False:

T a. If Gaussian elimination is used with NO pivoting, large roundoff errors may result even if A is well-conditioned.

F b. If Gaussian elimination is used with partial pivoting, the solution is usually very accurate even if A is ill-conditioned.

F c. The Gauss-Seidel iterative method (for $Ax = b$) is generally slower than the Jacobi method.

F d. The Jacobi iterative method (for $Ax = b$) converges only if the matrix is diagonal-dominant.

F e. A quadrature method which has $O(h^3)$ error will give a smaller error than an $O(h)$ method, for any h.

T f. Roundoff error is much more serious, in general, for derivative approximations than for integral approximations.

T g. Gaussian elimination, when applied to a general N by N linear system, requires $O(N^3)$ arithmetic operations.

F h. If $s(x)$ is a cubic spline, then s, s', s'' and s''' must be continuous everywhere.

F i. If a quadrature method is exact for all polynomials of degree n, its error is $O(h^n)$ for general smooth functions.

F j. If a matrix A has condition number 10, we expect to lose about 10 significant digits in solving $Ax = b$ with Gauss elimination and partial pivoting.

Math 4329, Test II (h)

Name Key

1. If

$$\begin{aligned} s(x) &= 0 & for\ 1 \le x \le 2 \\ s(x) &= A(x-2)^3 & for\ 2 \le x \le 3 \end{aligned}$$

a. For what value(s) of A is $s(x)$ a cubic spline?

3

$s(2)=0$

$s'(x) = 3A(x-2)^2 \quad s'(2)=0$

$s''(x) = 6A(x-2) \quad s''(2)=0$

any A

b. For what value(s) of A is $s(x)$ a natural cubic spline?

1

$A=0$

$6A(3-2)=0$

$A=0$

2. If $P_4(x)$ is the fourth degree polynomial that interpolates to $f(x) = sin(2x)$ at $x = 0, 0.1, 0.2, 0.3, 0.4$, find a reasonable bound on the error at $x = 0.35$.

4

$$|f(x)-P_4(x)| = \left|\frac{f^{V}(\xi)}{5!}(x-0)(x-0.1)(x-0.2)(x-0.3)(x-0.4)\right|$$

$$\le \left|\frac{2^5\cos(2\xi)}{120}(.35)(.25)(.15)(.05)(-.05)\right| \le 8.75\cdot10^{-6}$$

3. Use Taylor series expansions to determine the error in the approximation $u'''(t) \approx \frac{u(t+3h)-3u(t+2h)+3u(t+h)-u(t)}{h^3}$

$1 \quad u(t+3h) = u + 3hu' + \frac{(3h)^2}{2}u'' + \frac{(3h)^3}{6}u''' + \frac{(3h)^4}{24}u^{iv} \dots$

$-3 \quad u(t+2h) = u + 2hu' + \frac{(2h)^2}{2}u'' + \frac{(2h)^3}{6}u''' + \frac{(2h)^4}{24}u^{iv} \dots$

4 $\quad 3 \quad u(t+h) = u + hu' + \frac{h^2}{2}u'' + \frac{h^3}{6}u''' + \frac{h^4}{24}u^{iv} \dots$

$-1 \quad u(t) = u$

$\text{num} = \quad 0 \quad 0 \quad 0 \quad \frac{h^3}{6}u'''(6) + \frac{h^4}{24}u^{iv}(36) \dots$

$\frac{\text{num}}{h^3} = u'''(t) + \frac{3}{2}hu^{iv}$

4. Find A, B which make the approximation

$\int_0^h f(x)dx \approx Ahf(0) + Bhf(\frac{2h}{3})$

as high degree of precision as possible. With your choice of A, B, what is the degree of precision, and what is the order of the error (power of h that the global error is proportional to) in this approximation?

4

$\int_0^h f(x)\,dx = Ahf(0) + Bhf(\frac{2h}{3})$

$f=1 \quad h \overset{\checkmark}{=} Ah + Bh \qquad A + B = 1 \qquad A = \frac{1}{4}$

$f=x \quad \frac{1}{2}h^2 \overset{\checkmark}{=} Ah\cdot 0 + Bh\cdot\frac{2h}{3} \qquad \frac{2}{3}B = \frac{1}{2} \qquad B = \frac{3}{4}$

$f=x^2 \quad \frac{1}{3}h^3 \overset{\checkmark}{=} Ah\cdot 0^2 + Bh\left(\frac{2h}{3}\right)^2 = \frac{3}{4}h\,\frac{4h^2}{9} = \frac{1}{3}h^3$

$f=x^3 \quad \frac{1}{4}h^4 \not= Ah\,0^3 + Bh\left(\frac{2h}{3}\right)^3 = \frac{3}{4}h\,\frac{8h^3}{27} = \frac{2}{9}h^4$

$O(h^3)$

5. Consider the linear system:

$$\begin{bmatrix} 7 & 2 & 1 \\ 0 & 3 & 2 \\ 1 & -3 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ -3 \end{bmatrix}$$

a. Write out the equations for the Jacobi iterative method for solving this system (don't actually do any iterations).

2

$x_{n+1} = (5 - 2y^n - z^n)/7$

$y_{n+1} = (4 - 2z^n)/3$

$z_{n+1} = (-3 - x^n + 3y^n)/5$

b. Write out the equations for the Gauss-Seidel iterative method for solving this system.

1

$x_{n+1} = (5 - 2y^n - z^n)/7$

$y_{n+1} = (4 - 2z^n)/3$

$z_{n+1} = (-3 - x^{n+1} + 3y^{n+1})/5$

c. True or False: the Jacobi iterative method (5a) will converge for *any* starting vector (x_0, y_0, z_0). Give a reason for your answer.

1

True - A is diagonal dominant

d. Given that

$$A^{-1} = \begin{bmatrix} 0.1419 & -0.0878 & 0.0068 \\ 0.0135 & 0.2297 & -0.0946 \\ -0.0203 & 0.1554 & 0.1419 \end{bmatrix}$$

2

find the condition number of A (using L_∞ norm). If you were to solve the linear system above using Gaussian elimination with partial pivoting, would you expect serious roundoff errors?

$\text{Cond}(A) = \|A\|\|A^{-1}\| = 10 \cdot 0.3378 = 3.378$

no serious r/o problem

Math 4329, Test II (i)

Name Key

1. a. A table of values for $f(x)$ is:

x	$f(x)$
10	0.1000
11	0.0000
12	0.0000
13	0.0000

$$\frac{(x-11)(x-12)(x-13)}{(10-11)(10-12)(10-13)} 0.1$$

$$= 0.03125$$

3

Use cubic interpolation to estimate $f(10.5)$.

b. If it is known that $|f^{iv}(x)| < 0.05$ for all x, obtain a reasonable bound on the error in your estimate of $f(10.5)$.

3

$$\left|\frac{f^{iv}(\xi)}{4!}(x-10)(x-11)(x-12)(x-13)\right| \le \frac{0.05}{24}(0.5)(0.5)(1.5)(2.5)$$

$$= 0.00195$$

2. Use Taylor series expansions to determine the error in the approximation $u''(x) \approx \frac{u(x+2h)-2u(x+h)+u(x)}{h^2}$

3

$$\begin{aligned}
1 \quad & u(x+2h) = u + 2u'h + 4u''\frac{h^2}{2} + 8u'''\frac{h^3}{6} \cdots \\
-2 \quad & u(x+h) = u + u'h + u''\frac{h^2}{2} + u'''\frac{h^3}{6} \cdots \\
1 \quad & u(x) = u \\
\hline
& u''h^2 + u'''h^3 + \cdots
\end{aligned}$$

$$\frac{u''h^2 + u'''h^3 + \cdots}{h^2} = u''(x) + hu'''(x) + \cdots$$

error

3. a. Find A, B which make the approximation

$\int_0^h f(x)dx \approx Ahf(0) + Bhf(\frac{2}{3}h)$

as high order as possible.

$B = \frac{3}{4} \quad A = \frac{1}{4}$

3

$f=1 \qquad h = \int_0^h f(x)dx = Ah + Bh \qquad A+B=1$

$f=x \qquad \frac{h^2}{2} = \int_0^h x\,dx = Ah(0) + Bh\frac{2}{3}h \qquad \frac{2}{3}B = \frac{1}{2}$

$f=x^2 \qquad \frac{h^3}{3} = \int_0^h x^2 dx = Ah0^2 + Bh\left(\frac{2h}{3}\right)^2 \qquad \frac{4}{9}B = \frac{1}{3}$ ✓

$f=x^3 \qquad \frac{h^4}{4} = \int_0^h x^3 dx = Ah0^3 + Bh\left(\frac{2h}{3}\right)^3 \qquad \frac{8}{27}B = \frac{1}{4}$ NO

b. What is the order of the global error, for this A,B?

1

$O(h^3)$

4. (Note: you must do by hand and show your work.) Find the inverse of

3

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix},$$

$$\begin{bmatrix} 1 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 \\ 2 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix}$$

4

5. A quintic spline interpolant is a function which is a polynomial of degree five or less in each interval $(x_{i-1}, x_i), i = 1, ..., N$ and passes through the points $(x_i, y_i), i = 0, ..., N$ and is continuous and has continuous first, second, third and fourth derivatives.

   a. How many unknown coefficients need to be determined? (Hint: There are N intervals and the quintic has how many coefficients in each?)

   $6N$

   b. How many interpolation conditions are there? (Hint: There are two interpolation conditions for each interval.)

   $2N$

   c. How many continuity conditions are there? (Hint: s(x) is automatically continuous because of the interpolation conditions, so we only need to require that s', s'', s''', s^{iv} be continuous at each *interior* point—how many interior points are there?)

   $4(N-1)$

   d. If you add the number of interpolation conditions (part b) and continuity conditions (part c), does this equal the number of unknowns (part a)? If not, what needs to be done to make the quintic spline interpolation problem have a unique solution?

   $2N + 4(N-1) = 6N - 4$ need 4 end condition

Math 4329, Test II (j)

Name Key

1. a. A table of values for $f(x) = sin(\pi x)$ is:

x	$f(x)$
1.0	0.0
1.5	-1.0
2.0	0.0

Use quadratic interpolation to estimate $f(1.6)$.

3

$$p_2(x) = \frac{(x-1)(x-2)}{(1.5-1)(1.5-2)}(-1) = 4x^2 - 12x + 8$$

$$p_2(1.6) = \frac{0.6\,(-0.4)}{0.5\,(-0.5)}(-1) = -0.96$$

b. Use the Lagrange error formula to obtain a reasonable bound on the error in your estimate $p_2(1.6)$ of $f(1.6)$.

3

$$|f(x) - p_2(x)| = \left|\frac{f'''(\xi)}{3!}(x-1)(x-1.5)(x-2)\right| = \left|\frac{-\pi^3 \cos(\pi\xi)}{6}(0.6)(0.1)(-0.4)\right|$$

$x = 1.6$

$1 \le \xi \le 2$

$$\le \frac{\pi^3}{6}(0.024) = 0.124$$

c. Calculate the exact error $f(1.6) - p_2(1.6)$.

1

$$f(1.6) = \sin(1.6\pi) = -0.951$$

$$f(1.6) - p_2(1.6) = -0.951 - (-0.96) = 0.009$$

2. Use Taylor series expansions to determine the error in the approximation $u'(x) \approx \frac{3u(x)-4u(x-h)+u(x-2h)}{2h}$

$3 \quad u(x) = u$

$-4 \quad u(x-h) = u - hu' + \frac{h^2}{2}u'' - \frac{h^3}{6}u''' + \dots$

$1 \quad u(x-2h) = u - 2hu' + 4\frac{h^2}{2}u'' - 8\frac{h^3}{6}u''' + \dots$

3

$\frac{\text{numerator}}{2h} = \frac{2hu' - \frac{4}{6}h^3u''' + \dots}{2h} = u'(x) - \frac{h^2}{3}u'''(x) + \dots$

error

3. The following function is a cubic spline for what values of a, b, c?

$$
\begin{aligned}
s(x) &= 2x^3 + 3x^2 + 2x + 5 \quad for \quad 0 < x \leq 1 \\
&= x^3 + ax^2 + bx + c \quad for \quad 1 < x \leq 2
\end{aligned}
$$

3

	$(0,1)$		$(1,2)$
s	$2x^3+3x^2+2x+5$	$12 = 1 + a + b + c$	x^3+ax^2+bx+c
s'	$6x^2+6x+2$	$14 = 3 + 2a + b$	$3x^2+2ax+b$
s''	$12x+6$	$18 = 6 + 2a$	$6x+2a$

$a = 6$
$b = -1$
$c = 6$

$14 = 3 + 12 + b$
$12 = 1 + 6 - 1 + c$

4. a. Find A, B which make the approximation

$\int_0^h f(x)dx \approx Ahf(0.5h) + Bhf(0.8h)$

as high order as possible.

$A=1$
$B=0$

$f=1$ $\quad h = \int_0^h 1\,dx = Ah + Bh$ $\quad A+B=1$

3 $\quad f=x$ $\quad \frac{1}{2}h^2 = \int_0^h x\,dx = Ah(0.5h) + Bh(0.8h)$ $\quad 0.5A + 0.8B = 0.5$

$f=x^2$ $\quad \frac{1}{3}h^3 = \int_0^h x^2 dx = Ah(0.25h^2) + Bh(0.64h^2)$ $\quad 0.25A + 0.64B \neq \frac{1}{3}$

$d_{0,p} = 1$

b. What is the order of the global error, for this A,B?

1

$O(h^2)$

5. (Note: you must do by hand and show your work.) Find the inverse of

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix},$$

3

$$A^{-1} = \begin{bmatrix} -1 & 1 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ -1 & 1 & -1 & 2 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

Math 4329, Test II (K)

Name Key

Solve any 3 of the first 4 problems, plus problems 5 and 6. Clearly mark which problem is not to be graded.

1. Use Taylor series expansions to determine the error in the approximation $u'''(t) \approx \frac{u(t+3h)-3u(t+2h)+3u(t+h)-u(t)}{h^3}$. (Hint: expand out to the h^4 terms.)

5

$-1 \quad u(t) = u$

$3 \quad u(t+h) = u + hu' + \frac{h^2}{2}u'' + \frac{h^3}{6}u''' + \frac{h^4}{24}u^{iv} + ..$

$-3 \quad u(t+2h) = u + 2hu' + \frac{4h^2}{2}u'' + \frac{8h^3}{6}u''' + \frac{16h^4}{24}u^{iv} + ..$

$1 \quad u(t+3h) = u + 3hu' + \frac{9h^2}{2}u'' + \frac{27h^3}{6}u''' + \frac{81h^4}{24}u^{iv} + ..$

numerator $= 0 \quad 0 \quad 0 \quad \frac{30-24}{6}h^3u''' + \frac{84-48}{24}h^4u^{iv} + ..$

$= h^3u''' + \frac{3}{2}h^4u^{iv} + ..$ $\quad$ rhs $= u'''(t) + \frac{3}{2}hu^{iv} + ..$

error

2. What is the degree of precision of the approximation:

$\int_0^h f(x)dx \approx \frac{2}{3}hf(\frac{1}{4}h) - \frac{1}{3}hf(\frac{2}{4}h) + \frac{2}{3}hf(\frac{3}{4}h)$

5

$f=1 \qquad h = \frac{2}{3}h - \frac{1}{3}h + \frac{2}{3}h = h$ ✓

$f=x \qquad \frac{1}{2}h^2 = \left(\frac{2}{3}\frac{1}{4} - \frac{1}{3}\frac{2}{4} + \frac{2}{3}\frac{3}{4}\right)h^2 = \frac{1}{2}h^2$ ✓

$f=x^2 \qquad \frac{1}{3}h^3 = \left(\frac{2}{3}\frac{1}{16} - \frac{1}{3}\frac{4}{16} + \frac{2}{3}\frac{9}{16}\right)h^3 = \frac{1}{3}h^3$ ✓

$f=x^3 \qquad \frac{1}{4}h^4 = \left(\frac{2}{3}\frac{1}{64} - \frac{1}{3}\frac{8}{64} + \frac{2}{3}\frac{27}{64}\right)h^4 = \frac{1}{4}h^4$ ✓

$f=x^4 \qquad \frac{1}{5}h^5 = \left(\frac{2}{3}\frac{1}{256} - \frac{1}{3}\frac{16}{256} + \frac{2}{3}\frac{81}{256}\right)h^5 = \frac{148}{3(256)}h^5 \neq 0.193$

dop = 3

3. A table of values for $f(x)$ is:

x	$f(x)$
100	0.0
110	3.0
120	0.0
130	1.0

Use cubic interpolation to estimate $f(125)$.

$$P_3(x) = \frac{(x-100)(x-120)(x-130)}{(110-100)(110-120)(110-130)}\,3 + \frac{(x-100)(x-110)(x-120)}{(130-100)(130-110)(130-120)}\,1$$

5

$$P_3(125) = \frac{(25)(5)(-5)}{(10)(-10)(-20)}\,3 + \frac{(25)(15)(5)}{(30)(20)(10)}\,1 = \frac{-3750}{6000} = -0.625$$

4. If it is known that $|f^{iv}(x)| < 0.001$ for all x, obtain a reasonable bound on the error in your estimate of $f(125)$ in problem 3.

$$-P_3(x)+f(x) = \left|\frac{f^{iv}(\xi)}{24}(x-100)(x-110)(x-120)(x-130)\right|$$

$$\leq \left|\frac{0.001}{24}(25)(15)(5)(-5)\right| = 0.3906$$

5

5. Define a natural cubic spline.

$S(x) = ax^3 + bx^2 + cx + d$ in $x_i \le x \le x_{i+1}$

S, S', S'' continuous everywhere

$S'' = 0$ at endpoints

3

6. Approximately how many multiplications are done by the section of MATLAB code below, which does back substitution in solving an N by N linear system?

3

```
X(N) = B(N)/A(N,N);
for I=N-1:-1:1
   SUM = 0.0;
   for J=I+1:N
      SUM = SUM + A(I,J)*X(J);
   end
   X(I) = (B(I)-SUM)/A(I,I);
end
```

$\frac{1}{2}N^2$

Math 4329, Test II (2)

Name Key

1. a. A table of values for $f(x)$ is:

x	$f(x)$
0.0	0.0
0.1	3.0
0.2	0.0

Use quadratic interpolation to estimate $f(0.05)$.

3

$$L_2(x) = \frac{(x-0)(x-0.2)}{(0.1-0)(0.1-0.2)} 3 \qquad L_2(0.05) = \frac{(0.05)(-0.15)}{0.1\ (-0.1)} 3 = 2.25$$

b. If $f(x) = 3\ sin(5\pi x)$, obtain a reasonable bound on the error in your estimate of $f(0.05)$.

3

$$\frac{(x-0)(x-0.1)(x-0.2)}{6} f'''(\eta) = \left|\frac{(.05)(-.05)(-.15)}{6} 3(5\pi)^3 \cos(5\pi\eta)\right|$$

$$\leq \frac{(.05)(.05)(.15)}{6} 3(5\pi)^3 = 0.726$$

2. Use Taylor series expansions to determine the error in the approximation $u''(t) \approx \frac{u(t)-2u(t-h)+u(t-2h)}{h^2}$

3

$$\begin{array}{rl} 1 & u(t) = u \\ -2 & u(t-h) = u - hu' + \frac{h^2}{2}u'' - \frac{h^3}{6}u''' \\ 1 & u(t-2h) = u - 2hu' + \frac{4h^2}{2}u'' - \frac{8h^3}{6}u''' \end{array}$$

$$\text{num} = h^2u'' - h^3u''' + \ldots \qquad \text{error} = -hu''' + \ldots$$

$$\frac{\text{num}}{h^2} = u''(t) - hu'''(t) + \ldots$$

3. The following function is a cubic spline for what values of a, b, c?

$$\begin{aligned} s(x) &= 2x^3 - 3x^2 + 3x - 4 \quad for \quad 0 < x \leq 1 \\ &= x^3 + ax^2 + bx + c \quad for \quad 1 < x \leq 2 \end{aligned}$$

$a = 0$
$b = 0$
$c = -3$

$x = 1$

$s = 2x^3 - 3x^2 + 3x - 4$ $-2 = 1 + a + b + c$ $x^3 + ax^2 + bx + c$

3   $s' = 6x^2 - 6x + 3$ $3 = 3 + 2a + b$ $3x^2 + 2ax + b$

$s'' = 12x - 6$ $6 = 6 + 2a$ $6x + 2a$

4. Determine values for A, B, C which make

$$\int_0^h f(x)dx \approx Ahf(0) + Bhf(h/3) + Chf(h)$$

$d.o.p = 2 \quad O(h^3)$

3   as high order as possible. What is the degree of precision and what is the global order?

$h = \int_0^h 1\,dx = Ah + Bh + Ch$ $A + B + C = 1$ $A = 0$ $\frac{h^4}{4} = \int_0^h x^3 dx$

$\frac{h^2}{2} = \int_0^h x\,dx = \quad Bh(\frac{h}{3}) + Ch(h)$ $\frac{1}{3}B + C = \frac{1}{2}$ $B = \frac{3}{4}$ $\cancel{A}Bh(\frac{h}{3})^3$

$\frac{h^3}{3} = \int_0^h x^2 dx = \quad Bh(\frac{h}{3})^2 + Ch\,h^2$ $\frac{1}{9}B + C = \frac{1}{3}$ $C = \frac{1}{4}$ $+ Ch\,h^3 = \frac{5}{18}h$

5. a. If a quadrature rule yields errors of 0.0064 when $h = 0.01$ and 0.0002 when $h = 0.0025$, what is the experimental order? ($O(h^?)$) $\alpha = 2.5$

5   b. Of all quadrature rules with n sample points per strip, the one with highest degree of precision is called what?   Gaussian n-point formula

c. Gaussian elimination, when applied to a general N by N linear system, requires approximately how many multiplications? $\frac{1}{3}N^3$

d. The strategy of switching rows during Gaussian elimination to always bring the largest (in absolute value) potential pivot to the diagonal is called what?   partial pivoting

e. True or False: If $f(x)$ is a smooth function and $L_n(x)$ is the Lagrange polynomial that interpolates to f at uniformly spaced points $a = x_0, x_1, ...x_n = b$, then for all $a < x < b$, $L_n(x)$ is guaranteed to converge to $f(x)$ as $n \to \infty$. (Assume no roundoff error.)   False

Math 4329, Test II (0)

Name Key

1. a. A table of values for $f(x)$ is:

x	$f(x)$
1.0	0.1
1.1	0.0
1.2	0.0
1.3	0.0

Use cubic interpolation to estimate $f(1.05)$.

2

$P_3(x) = \frac{(x-1.1)(x-1.2)(x-1.3)}{(1-1.1)(1-1.2)(1-1.3)} 0.1 \quad P_3(1.05) = \frac{(-0.05)(-0.15)(-0.25)}{(-0.1)(-0.2)(-0.3)} 0.1$

$= 0.03125$

b. If it is known that $|f^{iv}(x)| < 0.001$ for all $1 < x < 1.3$, obtain as small a bound as possible on the error in your estimate of $f(1.05)$.

2 $\quad |P_3(x) - f(x)| = \left| \frac{f^{IV}(c)}{4!}(x-1)(x-1.1)(x-1.2)(x-1.3)\right| \le \frac{.001}{24}(.05)(.05)(.15)(.25)$

$\approx 3.91 \cdot 10^{-9}$

2. Use Taylor series expansions to determine the error (not just the order of the error) in the approximation $u''(t) \approx \frac{u(t+h)-2u(t)+u(t-h)}{h^2}$

$u(t+h) = u + u'h + u''\frac{h^2}{2} + u'''\frac{h^3}{6} + u^{IV}\frac{h^4}{24} + \ldots$

3 $\quad u(t) = u$

$u(t-h) = u - u'h + u''\frac{h^2}{2} - u'''\frac{h^3}{6} + u^{IV}\frac{h^4}{24} + \ldots$

$\text{r.h.s} = \frac{u''h^2 + 2u^{IV}\frac{h^4}{24} + \ldots}{h^2} = u''(t) + u^{IV}(\xi)\frac{h^2}{12}$

3. Determine the value for r which makes

$\int_0^h f(x)dx \approx \frac{1}{3}hf(rh) + \frac{1}{3}hf(\frac{h}{2}) + \frac{1}{3}hf((1-r)h)$

as high order as possible. Is this the Gauss 3 point formula? no

$$h = \int_0^h 1\, dx = \frac{1}{3}h + \frac{1}{3}h + \frac{1}{3}h = h$$

$$\frac{h^2}{2} = \int_0^h x\,dx = \frac{1}{3}h\,rh + \frac{1}{3}h\frac{h}{2} + \frac{1}{3}h(1-r)h = \frac{1}{2}h^2$$

4

$$\frac{h^3}{3} = \int_0^h x^2 dx = \frac{1}{3}hr^2h^2 + \frac{1}{3}h\frac{h^2}{4} + \frac{1}{3}h(1-r)^2h^2$$

$2r^2 - 2r + \frac{5}{4} = 1$

$$= \frac{1}{3}h^3\left(r^2 + \frac{1}{4} + 1 - 2r + r^2\right) = \frac{1}{3}h^3\left(2r^2 - 2r + \frac{5}{4}\right)$$

$$r^2 - r + \frac{1}{8} = 0 \qquad r = \frac{1 - \sqrt{\frac{1}{2}}}{2} = 0.1464466$$

4. (a) If we fit a polynomial of degree N through N+1 points, then change the y value at one point, where does the polynomial change, in general? (b) What about a natural cubic spline interpolant, same question?

2

a) everywhere
b) everywhere

5. (a) What boundary conditions does a natural cubic spline satisfy? (b) Give a scenario where this spline is a better idea than a "not-a-knot" cubic spline. (c) Give a scenario where it is not as good.

3

a) $s'' = 0$ at end points
b) want the least wiggly smooth curve through (x_i, y_i)
c) want an accurate approximation to a general $f(x)$

6. Below is a MATLAB program to solve Ax=b with no pivoting

```
function x = geln(A,b,n)
% forward elimination
for k=1:n-1
   if abs(A(k,k)) == 0
      error('Zero pivot encountered')
   end
   for i=k+1:n
      amul = -A(i,k)/A(k,k);
% add amul times row k to row i
      for j=k:n
         A(i,j) = A(i,j) + amul*A(k,j);
      end
      b(i) = b(i) + amul*b(k);
   end
end
% back substitution
if A(n,n) == 0
   error('Zero pivot encountered')
end
x(n) = b(n)/A(n,n);
for k=n-1:-1:1
   sum = 0;
   for j=k+1:n
      sum = sum + A(k,j)*x(j);
   end
   x(k) = (b(k)-sum)/A(k,k);
end
```

4

(a) Indicate which line in this program will account for nearly 100% of the computer time, when n is large. (b) Approximately how many multiplications are done during the forward elimination? (c) What about the back substitution, same question? (d) Approximately how much would the computer time increase if pivoting were done? (Assume n is large still.)

b) $\frac{1}{3}n^3$

c) $\frac{1}{2}n^2$

d) hardly any increase