Math 4329, Final

Name ______________________________

1. If the second order Taylor series method (one more term than Euler's method) is used to solve $u' = t^2(1+u^2)$, write u_{n+1} in terms of h, t_n and u_n only. ($t_n = nh, u_n \approx u(t_n)$)

2. a. Let $T_4(x)$ be the Taylor polynomial of degree 4 which matches $f(x), f'(x), f''(x), f'''(x)$ and $f^{iv}(x)$ at $a = -0.1$, where $f(x) = x^6 + x^3$. Use the Taylor remainder formula to find a reasonable bound on

   $|T_4(0) - f(0)| \leq$

   b. Let $L_4(x)$ be the Lagrange polynomial of degree 4 which matches $f(x)$ at $x = -0.1, 0.1, 0.2, 0.3$ and 0.4, where $f(x) = x^6 + x^3$. Use the Lagrange error formula to find a reasonable bound on

   $|L_4(0) - f(0)| \leq$

3. a. A rootfinder produces consecutive errors of $0.01, 0.0003, 0.000001$. Estimate the order of the method.

b. A quadrature method produces estimates of an integral of 5.51 when $h = 0.1$ and 5.50007, when $h = 0.01$, and the exact integral is 5.5. Estimate the order of the method.

4. If

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}, A^{-1} = \begin{bmatrix} -1 & 1 & 0 & 1 \\ 1 & 0 & 0 & -1 \\ -1 & 1 & -1 & 2 \\ 1 & -1 & 1 & -1 \end{bmatrix},$$

a. Calculate the condition number of A.

b. Estimate the smallest (in absolute value) eigenvalue of A, and the corresponding eigenvector, using the inverse power iteration. Start with $x_0 =< 1, 1, 1, 1 >$ and do 4 iterations.

c. Do one iteration of Newton's method, starting from $(0, 0, 0, 0)$ to solve:
$f1(x1, x2, x3, x4) = x2 + x3 + x4 = 0$
$f2(x1, x2, x3, x4) = x1 + x2 - 1 = 0$
$f3(x1, x2, x3, x4) = x1 + x4 = 0$
$f4(x1, x2, x3, x4) = x3 + x4 = 0$
(Hint: notice that the Jacobian matrix is just A.)

5. How should A, B, r be chosen to make the approximation:

$\int_{-1}^{1} f(x)dx \approx Af(-r) + Bf(0) + Af(r)$

as high degree of precision as possible?

6. Write $\frac{\sqrt{4+x}-2}{x}$ in a form where there is no serious problem with roundoff, when $x \approx 0$.

7. a. Write the third order differential equation $u''' - 3u'' - u^3 = e^t$ as a system of three first order equations, that is, in the form:

$$
\begin{aligned}
u' &= f(t,u,v,w) = \\
v' &= g(t,u,v,w) = \\
w' &= h(t,u,v,w) =
\end{aligned}
$$

   b. Now write out the formulas for $u_{n+1}, v_{n+1}, w_{n+1}$ for Euler's method applied to this system of first order equations:

$$
\begin{aligned}
u_{n+1} &= \\
v_{n+1} &= \\
w_{n+1} &=
\end{aligned}
$$

8. Will the iteration $x_{n+1} = 4\,x_n(1-x_n)$ converge to the root 0.75, if the starting guess is sufficiently good? **Justify** your answer.