KNOTS OBTAINED BY TWISTING UNKNOTS

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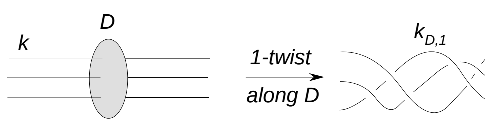
Abstract. Let K be the unknot in the 3-sphere S^3, and D a disk in S^3 meeting K transversely in the interior, at least twice (after all isotopies). We denote by $K_{D,n}$ a knot obtained from K by n twistings along the disk D. We describe for which pairs (K, D) and integers n, $K_{D,n}$ is a torus knot, a satellite knot or a hyperbolic knot.

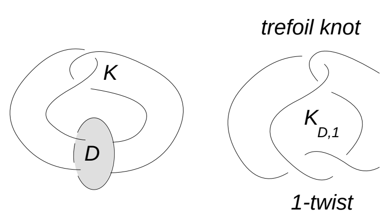
1. Twisted knots

1.1. **Definitions.** Let k a knot in S^3, and D a disk such that k intersects D transversely in its interior at least twice, after all isotopies of k in $S^3 - \partial D$. Let $k_{D,n}$ be the new knot obtained from k by performing n Dehn twists along D; $n = 1$ on the figure below.



By Ohyama [16], each knot can be obtained from a trivial knot by twisting along at most two properly chosen disks.

When $k = K$ is the unknot, then $K_{D,n}$ is said to be *a twisted knot*, and (K, D) *a twisting pair*. As an example, the trefoil knot is a twisted knot (see the figure below).



1.2. **Geometric types of knots.** Let k a knot in S^3, we denote by $E(k)$ the exterior of k : $E(k) = S^3 - \text{int}N(k)$, where $N(k)$ is a tubular neighbourhood of k in S^3.

By the Thurston's uniformization theorem [19], and the torus decomposition of Jaco, Shalen, Johannson [10, 11], $E(k)$ is either (1) a Seifert fibered space, (2) *toroidal*, i.e. contains an essential 2-torus (non-boundary-parallel and incompressible), or (3) hyperbolic (admits a complete hyperbolic structure of finite volume).

The knot k is respectively (1) a torus knot, (2) a satellite knot, or (3) a hyperbolic knot.

This is referred to be the *geometric type* of a knot in S^3. We are focus on the following question.

Question 1. *Can we have a good description of the geometric types of twisted knots ?*

In this context, one first question was :

Question 2 (Mathieu [13])**.** *Is there a composite twisted knot ?*

There exist composite twisted knots (see [5, 18]) but $n = \pm 1$.

Theorem 1 (Goodman-Strauss [5], Hayashi and Motegi [9])**.** *If a twisted knot $K_{D,n}$ is not prime then $n = \pm 1$.*

1.3. **Geometric types of twisting pairs.** We define the geometric type of a twisting pair in a similar way as for a knot in S^3. Let $E(K, C)$ be the exterior of $K \cup C$ in S^3, where $C = \partial D$: $E(K, C) = S^3 - \mathrm{int}N(K \cup C)$. Since $E(K, C)$ is irreducible and boundary-irreducible, by the Thurston's uniformization theorem [19], and the torus decomposition of Jaco, Shalen, Johannson [10, 11], $E(K, C)$ is either (1) a Seifert fibered space, (2) toroidal, or (3) hyperbolic.

We say that the twisting pair is respectively (1) Seifert fibered, (2) toroidal, or (3) hyperbolic.

This is referred to be the *geometric type* of a twisting pair in S^3. We can reformulate the fisrt question.

Question 3. *For which twisting pairs (K, D) and integers n, a twisted knot $K_{D,n}$ is a torus knot, a satellite knot or a hyperbolic knot ?*

Theorem 2. *If $\mid n \mid > 1$ then the twisted knot $K_{D,n}$ has the geometric type of the twisting pair (K, D).*

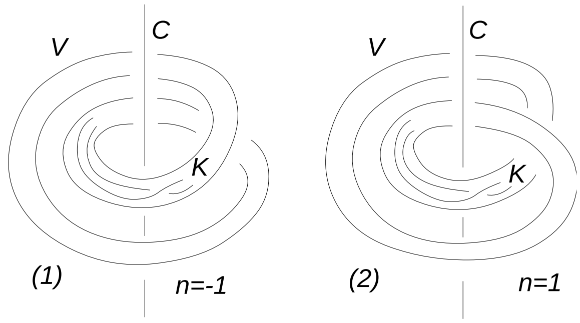
Section 2 is devoted to the Seifert fibered and toroidal twisting pairs; in particular, to the two following results.

Proposition 1. *If $E(K, C)$ is Seifert fibered, then $K_{D,n}$ is a torus knot for any integer n.*

Theorem 3. *If $E(K, C)$ is toroidal and $K_{D,n}$ is not a satellite knot, then :*

(i) (K, D) is a twisting pair as described in (1) *or* (2) *on the figure below, with the corresponding integer (so $\mid n \mid = 1$);*

(ii) Moreover, $V - \mathrm{int}N(K)$ is Seifert fibered or hyperbolic, and the twisted knot is a torus knot or hyperbolic knot, respectively.



If the twisting pair is hyperbolic, then the proof of Theorem 2 follows by the following proposition.

Proposition 2. *Assume that $E(K, C)$ is hyperbolic and $\mid n \mid > 1$. If $K_{D,n}$ is a satellite knot then $K_{D,n}$ is a cable of a torus knot.*

Section 3 is devoted to give the sketch of the proof of Proposition 2.

Proof of Theorem 2 when the twisting pair is hyperbolic. Let (K, D) be a hyperbolic twisting pair. Assume that $\mid n \mid > 1$.

By [14], $K_{D,n}$ is not a torus knot. Furthermore, by [1] the Gromov volume of $K_{D,n}$ is positive, i.e. $K_{D,n}$ is not a graph knot. Thus, by Proposition 2 $K_{D,n}$ is not satellite, so this is a hyperbolic knot. □

There exist non-hyperbolic twisted knots which come from hyperbolic twisting pairs. Here are some examples of such knots, among torus knots and satellite knots.

Torus knots from hyperbolic twisting pairs. The trefoil knot (see the second figure) is such a knot. There exist other examples by Goda, Hayashi and Song [4], and Wu [20].

Satellite knots from hyperbolic twisting pairs Until now, only two types of satellite twisted knots from hyperbolic twisting pair are known : composite knot (Motegi and Shibuya [15], see [2, Figure 1.3 (1)]) and cable of torus knot (Goda, Hayashi and Song [4], and Ohyama [16]).

Question 4. *Let (K, D) be a hyperbolic twisting pair. If $K_{D,n}$ is satellite, then is it a connected sum of a torus knot and some prime knot, or a cable of a torus knot ?*

For details concerning the results on this note, see [2, 3]. An alternative proof of Proposition 2 is given by Gordon and Luecke [8, Appendix A.2] in a more general setting.

2. NON-HYPERBOLIC TWISTING PAIRS

We consider successively that the twisting pair (K, D) is Seifert fibered and toroidal.

2.1. **Proof of Proposition 1.** If $E(K,C)$ is a Seifert fibered space, then $E(C)$ is a (p,q)-fibered solid torus, in which K is a regular fiber, and $p=1$ (K is unknotted). Therefore, $K_{D,n}$ is a $(1+nq,q)$-torus knot in S^3. □

2.2. **Sketch of the proof of Theorem 3.** Let T be an essential 2-torus in $E(K,C)$, where $C=\partial D$.

If $K_{D,n}$ is not satellite then T separates $\partial N(K)$ and $\partial N(C)$. Indeed, if T does not separate $\partial N(K)$ and $\partial N(C)$, then T bounds a solid torus V, which contains $K\cup C$; furthermore V is knotted in S^3. There is a 3-ball B_K (resp. B_C) in V which contains K (resp. C) but no 3-ball which contains $K\cup C$. Then $wind_V(K_{D,n}) = wind_V(K) = 0$ (algebraic intersection number with a meridian disk of V) so $K_{D,n}$ is not the core of V. If $K_{D,n}$ lies in a 3-ball in V, by Scharlemann results [17] about reducing Dehn surgeries on knots in solid tori, we get that C is a cable and that $\mid \frac{-1}{n} \mid$ is an integer bigger than one; a contradiction. Therefore, $K_{D,n}$ is a satellite knot.

Thus, we may assume that T separates $\partial N(K)$ and $\partial N(C)$: $S^3 = V\cup_T W$, where V,W are unknotted solid tori and $K\subset V$, $C\subset W$. Let ℓ be the core of V; note that $\mid \ell\cap D \mid\geq 2$ (because T is not parallel to $\partial N(C)$).

Therefore, (ℓ, D) is a twisting pair. If $\ell_{D,n}$ is knotted then $K_{D,n}$ is a satellite knot with companion knot $\ell_{D,n}$. By [13] or [12], if $\ell_{D,n}$ is unknotted then the twisting pair is described on the figure above, with the corresponding integer n.

Now, we note that :

(1) if $\mid n \mid > 1$ then $K_{D,n}$ is a satellite knot, and
(2) $K_{D,1} = K_{D_V,4}$ and $K_{D,-1} = K_{D_V,-4}$, where D_V is a meridian disk of V.

The geometric type of $V-\mathrm{int}N(K)$ is the one of the twisting pair (K, D_V); therefore (ii) follows by (1) above, Proposition 1 and Theorem 2 for the hyperbolic twisting pairs. □

3. Sketch of the proof of Proposition 2

We need to recall some basic definitions and properties about Dehn surgeries on knots and twisted knots.

3.1. **Dehn surgeries and twisted knots.** Let k be a knot in S^3, and α *a slope* (isotopy class of unoriented simple closed curve) on $\partial E(k)$. A α-*Dehn surgery on* k is gluing a solid torus V to $E(k)$ in such a way that a meridian disk of V is attached to $E(k)$ along the slope α : $k(\alpha) = E(k)(\alpha) = E(k) \bigcup_{\alpha=m} (S^1\times D^2)$, where m is the boundary slope of a meridian disk of V.

The slopes are parametrized by $\mathbb{Q}\cup\{\infty\}$ as usual. A slope α is denoted by $\frac{p}{q}$ if $\alpha\equiv p\mu+q\lambda$ in $H_1(\partial E(k))$, where μ is a meridian and λ a prefered longitude of k; note that (μ,λ) is a basis of $H_1(\partial E(k))$.

Let (K, D) be a twisting pair, and $C = \partial D$. We denote by M_n the 3-manifold obtained by a $\frac{-1}{n}$-Dehn surgery on C in $E(K)$, where n is an integer.

The twisted knot $K_{D,n}$ is the image of K after the $-\frac{1}{n}$-Dehn surgery on C; thus $M_n \cong E(K_{D,n})$. We denote by C_n the core of the attached solid torus.

Note that $M_0 = E(K) \cong S^1 \times D^2$ and $C_0 = C$.

3.2. **Finding punctured surfaces.** Let $\widehat{m}_D$ be a meridian disk of $M_0 = E(K) \cong S^1 \times D^2$. Isotope $\widehat{m}_D$ so that $d = \# \mid \widehat{m}_D \cap C_0 \mid$ is minimal. Since $E(K, C)$ is hyperbolic, we may note that $d \geq 2$.

Let m_D be the punctured disk $\widehat{m}_D \cap E(K, C)$. By the minimality of d, m_D is an essential surface in $E(K, C)$.

Let $\widehat{T}$ be an essential 2-torus in $E(K_{D,n})$, such that $t = \# \mid \widehat{T} \cap C_n \mid$ is minimal. Since $E(K, C)$ is hyperbolic $t \neq 0$. Furthermore, $\widehat{T}$ is separating so t is an even integer : $t \geq 2$. Let T be the punctured 2-torus $\widehat{T} \cap E(K, C)$. By the minimality of t, T is an essential surface in $E(K, C)$.

3.3. **Intersection graphs.** Let $(G_D, G_T) \subset (\widehat{m}_D, \widehat{T})$ be a pair of intersection graphs, defined as follows.

The (fat) vertices of G_D are the disk-components of $\widehat{m}_D - \text{int} m_D$ in $\widehat{m}_D$; similarly, the vertices of G_T are the disk-components of $\widehat{T} - \text{int} T$ in $\widehat{T}$. By an isotopy, we may assume that m_D and T are transverse and in general position in $E(K, C)$. Their intersection is the union of circles and arc-components. *The edges* of G_D (respectively G_T) are the arc-components of $m_D \cap T$ in $\widehat{m}_D$ (respectively $\widehat{T}$). For convenience, we often made the confusion between the arc-components of $m_D \cap T$, and the corresponding edges in both graphs.

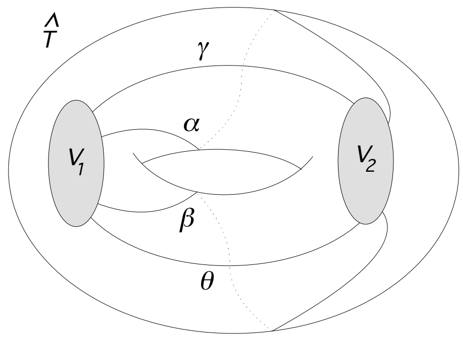
Assume that $\mid n \mid \geq 2$. Note that

- $\mid n \mid = \Delta(\frac{1}{0}, \frac{-1}{n})$, where $\Delta(\frac{1}{0}, \frac{-1}{n})$ denotes the distance (minimal geometric intersection number) between the slopes;
- $E(K, C)$ is hyperbolic;
- $\frac{1}{0}$ is a boundary reducing slope;
- $\frac{-1}{n}$ is a toroidal slope.

By the results of Gordon and Luecke [7] about the distance between a reducing slope and a toroidal slope on a hyperbolic manifold, we get that : $\mid n \mid = 2$ and $t = 2$.

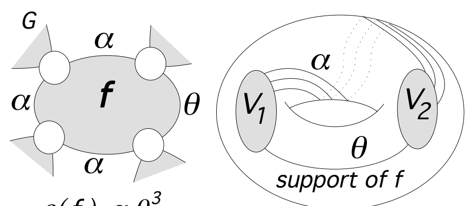
3.4. **Edge class labels.** Since G_T contains exactly two vertices V_1 and V_2 say, there are at most four *edge classes* (isotopy classes up to homeomorphism of $\widehat{T}$) of edges in G_T, which join V_1 to V_2; see the figure below. We call $\alpha, \beta, \theta, \gamma$ the *edge class labels*.

Let G be the subgraph of G_D, with all the vertices and all the corresponding edges in G_T that join V_1 to V_2.



$\alpha,\beta,\theta,\gamma$
edge class labels

Let f be a face of G. *The support of* f is the subgraph of G_T consisting of $V_1 \cup V_2$ and the correspondiong edges on ∂f. Note that the support of a face lies in an annulus in $\widehat{T}$ if and only if its boundary contains at most two edge class labels; see the figure below.



$\rho(f)=\alpha\,\theta^3$

Let f be a disk face of G. We define $\rho(f)$ to be the sequence of edge class labels around ∂f (up to cyclic permutation) in the anticlockwise direction; see the figure above. If $\rho(f) = \mu^x\lambda$, for some $\mu, \lambda \in \{\alpha, \beta, \gamma, \theta\}$, (where $\mu \neq \lambda$ and $x \neq 0$) f is said to be *primitive*. The support of a primitive face lies in an annulus in $\widehat{T}$.

3.5. **Black and white faces.** The 2-torus $\widehat{T}$ separates S^3 into two components : $V \cong S^1 \times D^2$ and $S^3 - V$. We say that V is the *black side* and $S^3 - V$, the *white side*.

Since m_D and T are essential in $E(K, C)$, no circle component of $m_D \cap T$ bounds a disk in m_D (cut and paste arguments).

A face f of G_D is said to be *a black (resp. white) face* if f lies in V (resp. in $S^3 - V$). For convenience, in the following a face is considered to be a disk-face.

3.6. **Sketch of the proof of Proposition 2.** Since $\widehat{T}$ is essential in M_n, V is knotted and $K_{D,n} \subset \mathrm{int}V$. Let ℓ be the core of V (a companion of $K_{D,n}$).

By the Euler-Poincaré formula and a combinatorial analysis (see [6]) G contains a disk face f of length 2 or 3; so f is primitive and its support lies in an annulus in $\widehat{T}$. By the two following propositions, one part of Proposition 2 is satisfied according to f is black or white.

Proposition 3. *If G contains a black face with the support lies in an annulus in $\widehat{T}$, then $K_{D,n}$ is a non-trivial cable of ℓ.*

Proposition 4. *If G contains a white primitive face, then the companion knot ℓ is a non-trivial torus knot.*

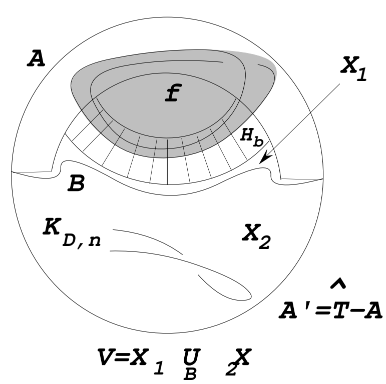
To complete the proof, we need another disk face g with the opposite colour of f. To be more precise, if f is white we need g to be black whose support lies in an annulus in $\widehat{T}$; if f is black we need g to be white and primitive. This is given by the two last propositions below.

Proposition 5.
i) All black faces of G are isomorphic, i.e. if g, h are black faces of G then $\rho(g) = \rho(h)$;
ii) If f is black then G contains a white primitive face.

Proposition 6.
i) G cannot contain two white primitive faces with exactly one edge class label in common.
ii) If f is white then G contains a black face, whose support lies in an annulus in $\widehat{T}$.

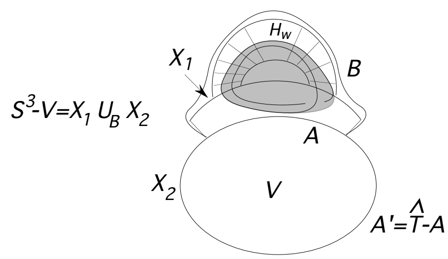
We finish by giving the ideas for proving these four propositions. We denote respectively by H_b, H_w the black and white handles, i.e. the intersection of the attached solid torus in M_n, with the black side V, or the white side $S^3 - V$.

3.6.1. *Proposition 3.* Let $X_1 = N(A \cup H_b \cup f)$ pushed slightly inside V. Note that ∂X_1 is a 2-torus, which contains A. Let B be the annulus $\partial X_1 - A$; see the figure below.



Since $K_{D,n}$ is not a composite knot (by [5, 9]) then A is not a meridonal annulus of V. Moreover, $E(K, C)$ is hyperbolic, so the annulus B is parallel to A'. Then X_2 is a solid torus and $K_{D,n}$ is a core of X_2. Therefore, $K_{D,n}$ is a non-trivial cable of ℓ.

3.6.2. *Proposition 4.* Let $X_1 = N(A \cup H_w \cup f)$ pushed slightly oustide V. Note that ∂X_1 is a 2-torus, which contains A. Let B be the annulus $\partial X_1 - A$; see the figure below.



Assume that f is primitive, i.e. $\rho(f) = \mu^x\lambda$ for some edge class labels μ, λ (with $\mu \neq \lambda$ and x a positive integer). Note that $N(A) \cup H_w$ is a genus 2 handlebody. Let m_1 be the co-core of H_w and m_2 be a meridian disk of $N(A)$. Then $\# \mid \partial f \cap m_2 \mid = 1$ so ∂f is a primitive curve on $\partial(N(A) \cup H_w)$ (this is the reason we choose 'primitive'as a word for such faces) and X_1 is a solid torus.

Since $K_{D,n}$ is not a composite knot and $E(K,C)$ is hyperbolic, X_2 is also a solid torus.

Finally, V is a knotted solid torus, so $S^3 - V \ncong S^1 \times D^2$ and $S^3 - V = X_1 \cup_B X_2$; thus $S^3 - V$ is a Seifert fibered space over a disk with two exceptional fibers, and ℓ is a non-trivial torus knot.

3.6.3. *Proposition 5.* (i) Let $W = \overline{V - \mathrm{int}N(K_{D,n}) - H_b}$; then $\partial W = \partial W_+ \coprod \partial W_-$, where $\partial W_+ = T \cup (\partial H_b - \widehat{T})$ and $\partial W_- = \partial N(K_{D,n})$.

Let g a black disk face of G, then ∂g is a non-separating simple closed curve in ∂W_+.

Since $E(K,C)$ hyperbolic, $W \cong \partial N(K_{D,n}) \times [0,1] \cup N(f)$; so W is a compression body, and g is the unique non-separating disk on W up to isotopy. Therefore, if g, h are two black disk faces of G, then g and h are isotopic on W, so ∂g and ∂h are freely homotopic in $\widehat{T} \cup H_b$. Now, $\pi_1(\widehat{T} \cup H_b) \cong \pi_1(\widehat{T}) * \mathbb{Z} \Rightarrow \rho(f) = \rho(g)$.

(ii) By (i) we may assume that the black faces of G all are isomorphic bigons (disk faces of length two) or isomorphic trigons (disk faces of length three). We conclude by a combinatorial analysis of G with this observation.

3.6.4. *Proposition 6.* (i) Since f a white primitive face of G, there exists A_f an annulus in $\widehat{T}$, which contains the support of f; let a_f be the core of A_f. Similarly for g another white primitive face of G, A_g denotes an annulus in $\widehat{T}$, which contains the support of g and a_g denotes the core of A_g.

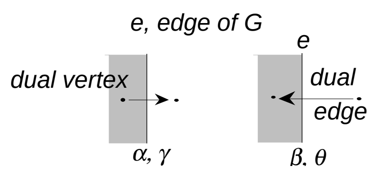
Since $S^3 - V$ is a Seifert fibered space over a disk with two exceptional fibers, $S^3 - V$ contains a unique properly embedded essential annulus (up to isotopy) A say. Note that the core of A has to be parallel to a_f and a_g. Therefore, $\# \mid a_f.a_g \mid = 1$ is impossible, which implies that ∂f and ∂g cannot have a single edge class label in common.

(ii) Since f is length two or three, we may interchange the edge class labels if necessary to get that $\rho(f) = \alpha\beta^n$, with $n = 1$ or 2.

Remark 1. *The graph G does not contain a face, with a single edge class label on its boundary.*

Proof. Let h be a face with a single edge class label on its boundary. Then its support lies in a disk B say in $\widehat{T}$. Let $X = N(B \cup h \cup H)$, where H is the black or white handle, according to h is black or white respectively. Then X is a punctured non-trivial lens space in S^3; a contradiction. □

Let Γ be a dual graph with oriented edges, defined as follows. We consider a dual vertex on each face of G. Now, for each edge e of G, we consider a dual edge transverse to e and joining the dual vertices on the opposite side of e. Moreover, we orient the dual edges from the black face to the white face if e is a α-edge or a γ-edge; otherwise we orient the dual edge in the opposite (from the white face to the black face); see the figure below.



Note that a sink (or source) of Γ is a face g of G such that :
(1) there are exactly (by Remark 1) two edge class labels on ∂g;
(2) ∂g and ∂f contain a single edge class label in common.

A combinatorial analysis leads to prove that Γ contains a sink (or source) which correspond to a black face or a white primitive face of G, g say. By (i) and (2) g is black, and by (1) its support lies in an annulus in $\widehat{T}$.

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