Seifert surfaces and twisting

MOHAMED AIT NOUH
Department of Mathematics,
California State University Channel Islands
One University Drive, Camarillo, CA 93102
e-mail: mohamed.aitnouh@csuci.edu

ABSTRACT

C. Mc. Gordon conjectured that there is no prime, twist reduced diagram of the unknot with at least 2 twist regions and at least 2 crossings per twist region. We use dimension four to prove Gordon's conjecture by the positive in case all the linking numbers are even and greater or equal to 4.

1. Introduction

Let K_n be the n-twisted knot obtained from K along n trivial circles $C_1, ..., C_t$.

Theorem 1.1. *$C_1, ..., C_t$ intersects some minimal genus Seifert surface for K_n always in the same direction.*

Conjecture 1.1 (Cameron C. Gordon)

If the twisting region of a reduced knot K has crossing number $c_i \geq 2$ for $i = 1, 2, .., n$. Then K is knotted.

We answer Conjecture 1 for $c_i \geq 2$ for $i = 1, 2, .., n$.

Let S be a Seifert surface for K_n that does intersect C always in the same direction. If the same direction, we can add handles as in Figure 1, but this woud increase the genus of S. Now consider $M = S^3 - intN(K_n)$ and take a sutured manifold decomposition

$$(M, \emptyset) \longrightarrow^{S_1} (M_1, \gamma_1) ... \longrightarrow^{S_n} (M_n, \gamma_n)$$

∂M_n is a union of spheres.

Lemma 2.1. (see [6], [3]) *If (M_n, γ_n) is $\emptyset$-taut then S is $\emptyset$-taut, i.e. S is of minimal genus*

Assume for a contradiction that (M_n, γ_n) is not taut. We consider the parametrizing surface (see [6] for definition) Q which is the punctured (by C) disk bounded by K. We can see that $\partial Q = K \bigcup_{i=1}^{i=q} C_i$ (see Figure 2); where $[\partial Q_i] = \frac{1}{n}$.
If (M_n, γ_n) is taut, then there exists a 2-sphere in ∂M_n which contains more than one suture (see ....). We take then the top disk D of this sphere. Consider now $D \cap D$. We have the following lemma:

Lemma 2.2. *Not many ($< q$) edges leave.*

Proof. This $\chi(Q) < 1 - q$. Then, the technics in Gabai's proof of Property R applies her (see [3]).

Corrolary 1 *If (M_n, γ_n) is not taut, then $\hat{Q}$ contains a Scharlemann cycle.*

Proof. By Lemma 2.2, let i be the end-point which does not leave. Then $\hat{Q}$ contains a great i-cycle (see Figure 4). This implies that $\hat{Q}$ contains a Scharlemann cycle. This is a contradiction. Consequentely, (M_n, γ_n) is taut, and then by Lemma 2.1 we have $g(S)$ is minimal.

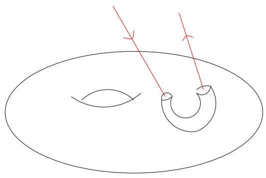

Figure 1:

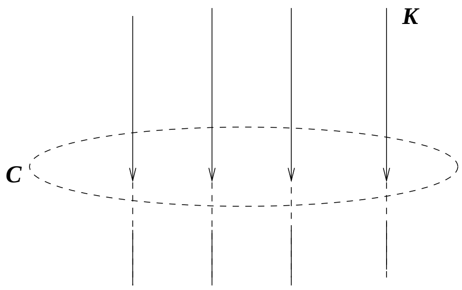

Figure 2:

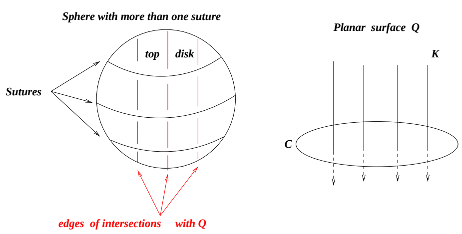

Figure 3:

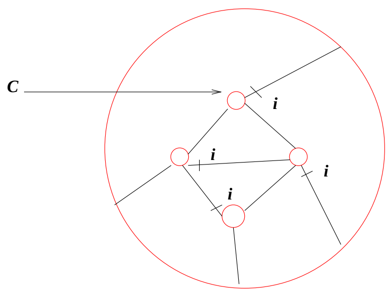

Figure 4:

Theorem 2.1. (D. Acosta [1], R. Fintushel[2], A. Yasuhara [8]) Let X be a smooth closed oriented simply connected 4-manifold with $m = min(b_2^+(X), b_2^-(X))$ and $M = max(b_2^+(X), b_2^-(X))$, and assume that $m \geq 2$. Suppose Σ is an embedded surface in X of genus g so that $[\Sigma]$ is characteristic. Then

$$g \geq \begin{cases} \dfrac{\mid \Sigma.\Sigma - \sigma(X) \mid}{8} + 2 - M & if \quad \Sigma.\Sigma \leq \sigma(X) \leq 0 \quad or \quad 0 \leq \sigma(X) \leq \Sigma.\Sigma \\ \dfrac{9(\mid \Sigma.\Sigma - \sigma(X) \mid)}{8} + 2 - M & if \quad \sigma(X) \leq \Sigma.\Sigma \leq 0 \quad or \quad 0 \leq \Sigma.\Sigma \leq \sigma(X) \\ \dfrac{\mid \Sigma.\Sigma - \sigma(X) \mid}{8} + 2 - m & if \quad \sigma(X) \leq 0 \leq \Sigma.\Sigma \quad or \quad \Sigma.\Sigma \leq 0 \leq \sigma(X) \end{cases}$$

Theorem 2.2. (R. Robertello) Let M be an oriented, compact 4-manifold with boundary a 3-sphere, and K a knot in ∂M. If K bounds a disk in M representing a characteristic element in $H_2(M, \partial M)$, then we have:

$$\frac{\sigma(M) - \xi.\xi}{8} \equiv Arf(K) \quad (mod.2)$$

Main Theorem

$$K \overset{(n_1,2\epsilon_1)}{\longrightarrow} K(n_1)... \overset{(n_p,2\epsilon_p)}{\longrightarrow} \overset{(n_1^{'},2\epsilon_1^{'})}{\longrightarrow} ... \overset{(n_n^{'},2\epsilon_n^{'})}{\longrightarrow} K(n_1, ..., n_k) = K_{D,n}.$$

With

(1) $\mid \epsilon_i \mid= 1$ and $\mid \epsilon_j^{'} \mid= 1$ for respectively $i = 1, 2, ..., p$ and $j = 1, 2, ..., n$.

(2) $n_i \geq 2$ and even for $i = 1, 2, ..., p$.

(3) $n_i^{'} \leq -2$ and even for $i = 1, 2, ..., n$.

(4) n_i and $n_i^{'}$ are even for respectively $i = 1, 2, ..., p$ and $i = 1, 2, ..., n$.

Proof of Main Theorem

By Theorem 1.1, we can assume that $\epsilon_i = \epsilon_n^{'} = 1$, or equivalentely

$$K \overset{(n_1,2)}{\longrightarrow} K(n_1)... \overset{(n_p,2)}{\longrightarrow} \overset{(n_1^{'},2)}{\longrightarrow} ... \overset{(n_n^{'},2)}{\longrightarrow} K(n_1, ..., n_k) = K_{D,n}.$$

The condition of the main theorem implies that there exists a properley embedded disk. It was proved in [5] and [**?**], using Kirby's calculus on the Hopf link [4], that this yields the existence of a properly embedded disk $D \subset S^2 \times S^2 - B^4$ such that

$$[D] = \sum_{i=1}^{i=p} -2\alpha_i + n_i\beta_i + \sum_{i=1}^{i=n} 2\alpha_i + n_i\beta_i$$

2000 Mathematics Subject Classification. 57M25, 57M45

Key Words and Phrases. Seifert surface, Smooth genus, Slice knot, torus knot, Tristram's signature, twisting, blow-up.

References

[1] Acosta, D., *Spin orbifolds and the minimal genus problem,* Dissertation, Tulane University, 1997.

[2] R. Fintushel, *Private communications with T. Lawson* (1997).

[3] D. Gabai, *Foliations and the topology of* 3*-manifolds, III* J. Diff. Geom.

[4] R. C. Kirby, The Topology of 4-manifolds, Lectures Notes in Mathematics, Springer-Verlag , 1980.

[5] K. Miyazaki and A. Yasuhara, *Knots that can not be obtained from a trivial knot by twisting,* Comtemporary Mathematics **164** (1994) 139-150.

[6] M. Scharlemann, *Sutured manifold and generalized Thurston norms,* 29 (1989) $557 - 614$.

[7] Eudave Munoz Mario, *Band sums of links which yields composite links. The cabling conjecture is true for strongly invertible knots* Trans. Amer. Math. Soc. 330(1992), no. 2, $463 - 501$.

[8] A. Yasuhara, *Connecting lemmas and representing homology classes of simply connected 4-manifolds,* Tokyo J. Math., **19** (1996), 245-261.

[9] A. Yasuhara, *On slice knots in the complex projective plane,* Rev. Mat. Univ. Complut. Madrid, **5** (1992), 255-276.