Twisting of positive knots

ABSTRACT

In this paper, we prove that a positive twisted knots is obtained with $n > 0$, except some special cases.

1. Introduction

Let K be a knot in the 3-sphere S^3, and D^2 a disk intersecting K in its interior. Let n be an integer. A $(-\frac{1}{n})$-Dehn surgery along $C = \partial D^2$ changes K into a new knot K_n in S^3. Let $\omega = \mathrm{lk}(\partial D^2, L)$. We say that K_n is obtained from K by (n, ω)-*twisting* (or simply *twisting*). Then we write $K \overset{(n,\omega)}{\rightarrow} K_n$, or $K \overset{(n,\omega)}{\rightarrow} K(n, \omega)$. We say that K_n is (n, ω)-twisted provided that K is the unknot (see Figure 2).

An easy example is depicted in Figure 2, where we show that the right-handed trefoil $T(2, 3)$ is obtained from the unknot $T(2, 1)$ by a $(+1, 2)$-twisting (In this case $n = +1$ and $\omega = +2$). Less obvious examples are given in Figure 6.

Definition 1.1. A knot k in S^3 is called a *positive knot* if every crossing of k is positive.

In this paper, we prove the following theorem regarding twisting of positive knots.

Theorem 1.1. *Let K be a positive knot and K is (n, ω)-twisted knot. Then*

- *Either $n > 0$, or*
- *$n < 0$ and $\omega = 0$ or ± 1. Furthemore, if $\omega = 0$, then $tau(K) \leq\mid n \mid$ and if $\omega = \pm 1$, then $Arf(K) = 0$.*

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2. **Preliminaries**

2.1. Embedding of surfaces in 4-manifolds:

In the following, $b_2^+(X)$ (resp. $b_2^-(X)$) is the rank of the positive (resp. negative) part of the intersection form of an oriented, compact 4-manifold X. Let $\sigma(X)$ denote the signature of X. Then a class $\xi \in H_2(X,\mathbb{Z})$ is said to be characteristic provided that $\xi.x \equiv x.x$ for any $x \in H_2(X,\mathbb{Z})$ where $\xi.x$ stands for the pairing of ξ and x, i.e. their Kronecker index and ξ^2 for the self-intersection of ξ in $H_2(X,\mathbb{Z})$. Let $\sigma_d(K)$ denotes the Tristram's signature of K [14], and let $Arf(K)$ denotes the Arf invariant of K.

The following theorem is originally due to O.Ya. Viro [15]. It is also obtained by letting $a = [d/2]$ in the inequality of [4, Remarks(a) on p-371] by P. Gilmer.

Theorem 2.1. Let X be an oriented, compact 4-manifold with $\partial X = S^3$, and K a knot in ∂X. Suppose K bounds a surface of genus g in X representing an element ξ in $H_2(X, \partial X)$.

(1) If ξ is divisible by an odd prime d, then: $\mid \dfrac{d^2-1}{2d^2}\xi^2 - \sigma(X) - \sigma_d(K) \mid \leq dim H_2(X;\mathbb{Z}_d) + 2g.$

(2) If ξ is divisible by 2, then: $\mid \dfrac{\xi^2}{2} - \sigma(X) - \sigma(K) \mid \leq dim H_2(X;\mathbb{Z}_2) + 2g.$

The following theorem is the definition Robertello's Arf invariant:

Theorem 2.1.

$$\frac{\xi^2 - \sigma(X)}{8} \equiv Arf(K) \quad (mod.8)$$

3. Proof of Theorem 1.1

Assume for a contradiction that K can be untied by (n,ω)-twisting along an unknot U. Assume that $n < 0$. Then k bounds a properly embedded smooth disk $(D, \partial D) \subset (\mid n \mid \mathbb{C}P^2 - B^4, \partial(\mid n \mid \mathbb{C}P^2 - B^4 \cong S^3)$ such that:

$$[D] = \omega(\bar{\gamma}_1 + ... + \bar{\gamma}_n) \in H_2(n\overline{\mathbb{C}P^2} - intB^4, S^3; \mathbb{Z}).$$

where $\bar{\gamma}_1, \bar{\gamma}_2, ..., \bar{\gamma}_n$ are the standard generators of $H_2(n\overline{\mathbb{C}P^2} - intB^4, S^3; \mathbb{Z})$ with the intersection number $\bar{\gamma}_i \cdot \bar{\gamma}_j = -\delta_{ij}$; where δ_{ij} is the Kronecker's delta.

Case 2.1. If ω is even, then Theorem 2.2 yields that $\mid \dfrac{\mid n \mid \omega^2}{2} - \mid n \mid - \sigma(k) \mid \leq \mid n \mid$. Or equivalently, $\mid\mid n \mid (\dfrac{\omega^2}{2} - 1) - \sigma(k) \mid \leq \mid n \mid$. This implies that

$$| n | (\frac{\omega^2}{2} - 1) \leq \sigma(k) \leq | n | \frac{\omega^2}{2}$$

This yields that $\omega = 0$.

Case 2.2. If $\omega \geq 3$ is odd, then let $d > 2$ denote the smallest prime divisor of ω. Gilmer-Viro's Theorem yields that $|| n | \omega^2 \frac{d^2-1}{2d^2} - | n | - \sigma_d(k) | \leq | n |$. Or equivalently,

$$| n | (\omega^2 \frac{d^2-1}{2d^2} - 2) \leq \sigma_d(k) \leq | n | \omega^2 \frac{d^2-1}{2d^2}$$

Since k is a positive knot, then $\sigma_d(k) < 0$, a contradiction.

Case 2.3. If $\omega = 1$, then by Robertello's Theorem, $Arf(k) = 0$.

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