TORUS KNOTS UNDER TWISTING

Mohamed AIT NOUH and Logan GODKIN

Abstract. In this paper, we give infinite family of torus knots that can not be untied by twisting.

1. Introduction

Let K be a knot in the 3-sphere S^3, and D^2 a disk intersecting K in its interior. Let $\omega = |\mathrm{lk}(\partial D^2, L)|$, and n integer. A $-1/n$-Dehn surgery along ∂D^2 changes K into a new knot K' in S^3. We say that K' is obtain from K by (n, ω)-*twisting* (or simply *twisting*). Then we write $K' \stackrel{(n,\omega)}{\rightarrow} K$. Let $\mathcal{T}$ denote the set of knots that are obtained from a trivial knot by a single twisting. Y. Ohyama [11] showed that any knot can be untied by two twistings.

A (p, q)-*torus knot* $T(p, q)$ is a knot that wraps around the standard solid torus in the longitudinal direction p times and the meridional direction q times, where the linking number of the meridian and lingitude is equlal to 1. Note that p and q are coprime. A torus knot $T(p, q)$ $(0 < p < q)$ is *exceptional* if $q \equiv \pm 1 \pmod p$, and *non-exceptional* if it is not exceptional.

Let $p(\geq 2)$ be an integer. It is not hard to see that $T(p, \pm 1) \stackrel{(k,p)}{\rightarrow} T(p, kp \pm 1)$. Since $T(p, \pm 1)$ is a trivial knot, $T(p, kp \pm 1)$ belongs to $\mathcal{T}$. This implies that any exceptional torus knot belongs to $\mathcal{T}$. In particular, all of the knots $T(2, q)$, $T(3, q)$, $T(4, q)$ and $T(6, q)$ belong to $\mathcal{T}$. In contrast with this fact, Goda-Hayashi-Song proved that $T(p, p + 2)$ does not belong to $\mathcal{T}$. This gave a counterexample to an old conjecture due to Ait Nouh and Yasuhara that states that any non-exceptional torus knot does not belong to $\mathcal{T}$.

These facts let us hit on the following

Conjecture. *Any non-exceptional (p, q)-torus knot, with $q \neq p + 2$, does not belong to $\mathcal{T}$.*

K. Miyazaki and A. Yasuhara [10] gave a sufficient condition for a knot not to be contained in $\mathcal{T}$ and showed that there are infinitely many knots that are not contained in $\mathcal{T}$. The first author and A. Yasuhara [2] proved that the family of $(p, p+4)$-torus knots is not contained in $\mathcal{T}$, and also proved that $T(5, 8)$ is the smallest torus knot not contained in $\mathcal{T}$.

Remark 1.1. In [9], K. Miyazaki and K. Motegi showed that if a non-exceptional torus knot $T(p, q)$ $(0 < p < q)$ is obtained from a trivial knot by a single (n, ω)-twisting, then $|n| = 1$. M. Ait Nouh and A. Yasuhara proved in [2] that $n = +1$.

AMS Classification numbers: 57Q25, 57Q45

Keywords: Torus Knots, twisting operations.

We would like to thank Professors C. Adams and L. Ludwig; the organizers of the Undergraduate Knot Theory Conference in Ohio held at Denison University, Granville July 15-17.

There are several restrictions on embedding of smooth surfaces into 4-manifolds. We use theorems from 4-dimensional topology to prove the following theorem:

Theorem 1.1 *Let p be an odd integer. If $p \geq 9$ and p is odd, then $T(p, p+6)$ does not belong to $\mathcal{T}$.*

2.Preliminaries:

2.1.Twisting operation and standard 4-manifolds:

There is a connection between twisting of knots in S^3 and dimension four: Any knot K_{-1} obtained from the unknot K (or more generally, a smooth slice knot in the 4-ball) by a $(-1, \omega)$-twisting is smoothly slice in $\mathbb{C}P^2$ with degree ω realizable by the twisting disk i.e. there exists a properly embedded smooth disk $\Delta \subset \mathbb{C}P^2 - B^4$ such that $\partial\Delta = K_{-1}$ and $[\Delta] = \omega\gamma \in H_2(\mathbb{C}P^2 - B^4, S^3, \mathbb{Z})$. For convenience of the reader, we give a sketch of a proof due that K. Miyazaki and A. Yasuhara [10]: We assume $K \cup C \subset \partial h^0 \cong S^3$, where h^0 denotes the 4-dimensional 0-handle ($h^0 \cong B^4$). The unknot K bounds a properly embedded smooth disk Δ in h^0. Then, performing a (-1)-twisting is equivalent to adding a 2-handle h^2, to h^0 along C with framing $+1$. It is known that the resulting 4-manifold $h^0 \cup h^2$ is $\mathbb{C}P^2 - B^4$ (see R. Kirby [7] for example). In addition, it is easy to verify that $[\Delta] = \omega\gamma \in H_2(\mathbb{C}P^2 - B^4, S^3, \mathbb{Z})$.

More generally, we can prove, using Kirby Calculus [7] and some twisting manipulations, that a (n, ω)-*twisted knot* in S^3 bounds a properly embedded smooth disk Δ in a punctured standard four manifold of the form $n\overline{\mathbb{C}P^2} - B^4$ if $n > 0$ (see Figure 3), or $\mid n \mid \mathbb{C}P^2 - B^4$ if $n < 0$. The second homology of $[\Delta]$ can be computed from n and ω. The disk Δ is called the *twisting disk*.

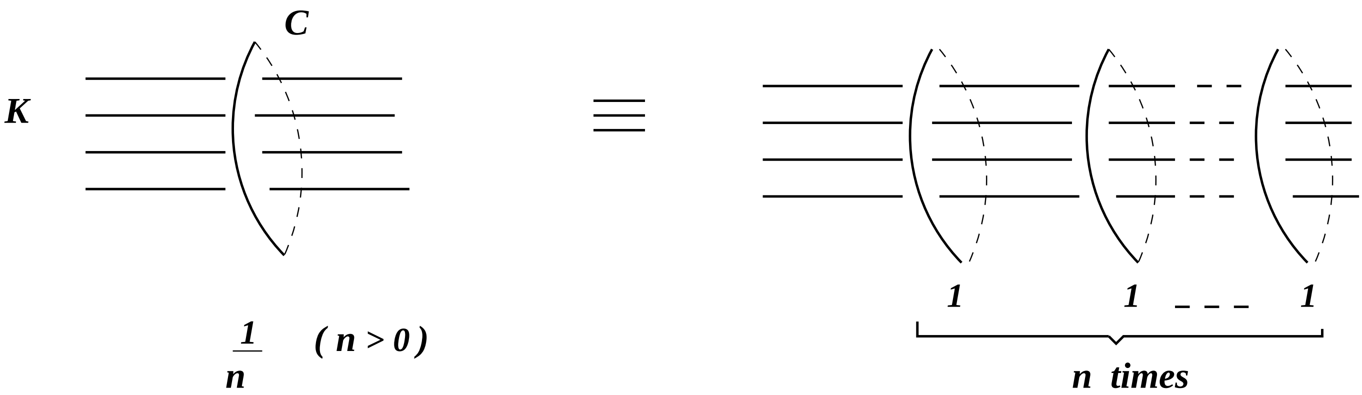

Figure 1:

2.2.Theorems from old gauge theory:

We use the following theorems from 4-dimensional topology to prove Theorem 1.1.In the following, let b_2^+ (resp. b_2^-) denote the dimension of the maximal positive (resp. negative) subspace for the intersection form on $H_2(X,\mathbb{Z})$. Let $\sigma(X)$ denote the signature of X, and denote by $\gamma_1,...,\gamma_n$ the standard generators of the free abelian group $H_2(X,\mathbb{Z})$. Then $\xi = \sum_{i=1}^{i=n} a_i\gamma_i \in H_2(X,\mathbb{Z})$ is said to be characteristic provided that $\xi.x \equiv x.x$ for any $x \in H_2(X,\mathbb{Z})$, where $\xi.x$ stands for the pairing of ξ and x, i.e. their Kronecker index. In particular, ξ^2 denotes the self-intersection of a class ξ in $H_2(X,\mathbb{Z})$.

Theorem 2.2.1. (P.M. Gilmer [3], O.Ya. Viro [14]) *Let M be a compact, oriented, once punctured 4-manifold, and K a knot in ∂M. Suppose that K bounds a properly embedded, oriented surface F in M that represent an element $\xi \in H_2(M,\partial M;\mathbb{Z})$.*

(1) *If ξ is divisible by an odd prime d, then:* $\mid \frac{d^2-1}{2d^2}\xi^2 - \sigma(X) - \sigma_d(K) \mid \leq dim H_2(X;\mathbb{Z}_d) + 2g.$

(2) *If ξ is divisible by 2, then:* $\mid \frac{\xi^2}{2} - \sigma(X) - \sigma(K) \mid \leq dim H_2(X;\mathbb{Z}_2) + 2g.$

Theorem 2.2.2 *(K. Kikuchi [6]) Let X^4 be a closed, oriented and smooth 4-manifold such that:*

- *$H_1(X^4)$ has no 2-torsion; and*
- *$b_2^{\pm 1} \leq 3$.*
  (Recall $\sigma(X^4) = b_2^+ - b_2^-$)
- *If $\xi = [S^2] \in H_2(X^4,\mathbb{Z})$ is a characteristic class then:*

$$\xi^2 = \sigma(X^4)$$

2.3. Signature of $(p, p+6)$-torus knots: To compute the signature of $(p, p+6)$-torus knot, we need the following proposition:

Proposition 2.3.1. (M. Ait Nouh and A. Yasuhara [2]) *Let $p(>0)$ be an odd integer and r $(2 \leq r < p)$ an even integer, and $T(p, p+r)$ a torus knot. Then*

$$\sigma(T(p,p+r)) = -\frac{(p-1)(p+r+1)}{2} + 2\sum_{i=1}^{r/2}\left(\left[\frac{(2i-1)p}{2r}\right] - \left[\frac{(2i-1)p+r}{2r}\right]\right)$$

Using Proposition 2.1, and some calculus, we have:

Proposition 2.3.2.

$$\sigma(T(p,p+6)) = \begin{cases} -\dfrac{(p-1)(p+7)}{2} & \textit{if } p \equiv 5 \quad (mod.12), \\ -\dfrac{(p-1)(p+7)}{2} - 6 & \textit{if } p \equiv 7 \textit{ or } 11 \quad (mod.12). \end{cases}$$

Proof.

By Proposition 2.3.1, we have

$$\sigma(T(p,p+6)) = -\frac{(p-1)(p+7)}{2} + 2\sum_{i=1}^{3}\left(\left[\frac{(2i-1)p}{12}\right] - \left[\frac{(2i-1)p+6}{12}\right]\right)$$

Which is equivalent to

$$\sigma(T(p,p+6)) = -\frac{(p-1)(p+7)}{2} + 2\left(\left[\frac{p}{12}\right] - \left[\frac{p+6}{12}\right]\right) + 2\left(\left[\frac{p}{4}\right] - \left[\frac{p+2}{4}\right]\right) + 2\left(\left[\frac{5p}{12}\right] - \left[\frac{5p+6}{12}\right]\right).$$

A straightforward arithmetics calculus yields Proposition 2.3.2.

3. Proofs of Theorems 1.1

Case 1. $p \equiv 5$ (mod. 12).

Assume for a contradiction that $T(p, p+6)$ is $(+1, \omega)$-twisted, then there exists a properly embedded disk $(\Delta, \partial\Delta) \subset (\overline{\mathbb{C}P^2} - B^4, S^3)$ such that $[\Delta] = \omega\bar{\gamma} \in H_2(\overline{\mathbb{C}P^2} - B^4, S^3)$. There are two subcases to consider according to ω is odd or even.

Case 1.1. If ω is odd, then notice that:

$$T(5,1) \cong U \overset{(1,5)}{\to} T(5,6) \cong T(6,5) \overset{(2n,6)}{\to} T(6,12n+5) \cong T(p,6) \overset{(1,p)}{\to} T(p,p+6).$$

Then the mirror-image $(p, p+6)$-torus knot can be obtained by the following twistings:

$$T(-5,1) \cong U \overset{(-1,5)}{\to} T(-5,6) \cong T(-6,5) \overset{(-2n,6)}{\to} T(-6,12n+5) \cong T(-p,6) \overset{(-1,p)}{\to} T(-p,p+6).$$

Therefore, there exists a properly embedded disk $(D, \partial D) \subset (\mathbb{C}P^2 \# S^2 \times S^2 \# \mathbb{C}P^2 - B^4, S^3)$ such that:

$$[D] = 5\gamma_1 + 6\alpha + 6n\beta + p\gamma_2 \in H_2(\mathbb{C}P^2 \# S^2 \times S^2 \# \mathbb{C}P^2 - B^4, S^3).$$

Assume that $T(p, p+6)$ is $(+1, \omega)$-twisted, then there exists a properly embedded disk $(\Delta, \partial\Delta) \subset (\overline{\mathbb{C}P^2} - B^4, S^3)$ such that $[\Delta] = \omega\bar{\gamma} \in H_2(\overline{\mathbb{C}P^2} - B^4, S^3)$. The sphere $[S^2] = [D \cup \Delta]$ satisfies:

$$[S^2] = 5\gamma_1 + 6\alpha + 6n\beta + p\gamma_2 + \omega\bar{\gamma} \in H_2(X^4, \mathbb{Z}).$$

Where $X^4 = \mathbb{C}P^2 \# S^2 \times S^2 \# \mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$. Since p and ω are odd, then $[S^2] \in H2(X^4, \mathbb{Z})$ is a characteristic class. This would contradicts Kikuchi's theorem. Note that $p = 12n + 5$ for some integer $n \geq 7$, then we would have:

$$\begin{aligned} [S^2].[S^2] = \sigma(X^4) \quad &\Longleftrightarrow \quad 25 + 2 \times 6 \times 6n + p^2 - \omega^2 = 1. \\ &\Longleftrightarrow \quad p^2 + 6p - 6 = \omega^2. \end{aligned}$$

$p^2 + 6p - 6$ is not a perfect square, a contradiction.

Case 1.2. If ω is even, then by Gilmer-Viro's theorem, we have

$$\mid -\frac{\omega^2}{2} - \sigma(T(p, p+6)) - \sigma(\overline{\mathbb{C}P^2}) \mid \leq 2.$$

or equivalently,

$$\frac{\omega^2}{2} - 3 \leq -\sigma \leq \frac{\omega^2}{2} + 1.$$

Which is in turns equivalent to

$$-\sigma(T(p, p+6)) = \frac{\omega^2}{2}.$$

or

$$-\sigma(T(p,p+6)) = \frac{\omega^2}{2} - 2.$$

By Proposition 2.2, $\sigma(T(p,p+6)) = -\dfrac{(p-1)(p+7)}{2}$ if $p \equiv 5 \pmod{12}$. This yields, that

$$(p-1)(p+7) = \omega^2.$$

or

$$(p-1)(p+7) - 4 = \omega^2.$$

It is easy to see that neither $(p-1)(p+7)$ nor $(p-1)(p+7)-4$ is a perfect square, by a discriminant argument.

Therefore, $T(p,p+6)$ is not twisted for any $p \geq 9$.

References

[1] **M. Ait-Nouh,** Les nœuds qui se dénouent par twist de Dehn dans la 3-sphère, Ph.D thesis, University of Provence, Marseille (France), (2000).

[2] **M. Ait-Nouh and A. Yasuhara,** Torus Knots that can not be untied by twisting, Revista Matemtica Complutense, XIV (2001), no. 8, 423-437.

[3] **P. Gilmer,** *Configurations of surfaces in 4-manifolds,* Trans. Amer. Math. Soc., **264** (1981), 353-38.

[4] **H. Goda and C. Hayashi and J. Song,** *Unknotted twistings of torus knots $T(p,p+2)$,* preprint (2003).

[5] **R. E. Gompf and Andras I. Stipsicz,** 4-manifolds and Kirby Calculus, *Graduate Studies in Mathematics*, Volume **20**, Amer. Math. Society. Providence, Rhode Island.

[6] **K. Kikuchi,** *Representing positive homology classes of* $\mathbb{C}P^2\#2\overline{\mathbb{C}P^2}$ *and* $\mathbb{C}P^2\#3\overline{\mathbb{C}P^2}$, Proc. Amer. Math. Soc. 117 (1993), no. 3, 861–869.

[7] **R. C. Kirby,** *The Topology of* 4*-manifolds*, Lectres Notes in Mathematics, Springer-Verlag , 1980.

[8] **R.A. Litherland,** Signatures on iterated torus knots, *Topology of low-dimensional manifolds (Proc. Second Sussex Conf., Chelwood Gate, 1977)*, 71-84, Lecture Notes in Math., **722**, Springer, Berlin, 1979.

[9] **K. Miyazaki and K. Motegi,** Seifert fibred manifolds and Dehn surgery, III, *Comme. Annal. Geom.*, **7** (1999), 551-582.

[10] **K. Miyazaki and A. Yasuhara,** Knots that can not be obtained from a trivial knot by twisting, *Comtemporary Mathematics* **164** (1994) 139-150.

[11] **Y. Ohyama,** *Twisting and unknotting operations,* Revista Math. Compl. Madrid, vol. 7 (1994), pp. 289-305.

[12] **D. Rolfsen,** Knots and Links, *Publish or Perish, Inc.* (1976).

[13] **A. Scorpan,** The wild world of 4-manifolds, *American Mathematical Society* (2005).

[14] **O. Ya Viro,** *Link types in codimension-*2 *with boundary,* Uspehi Mat. Nauk, **30** (1970), 231-232, (Russian).

[15] **A. Yasuhara,** $(2,15)$-torus Knot is not Slice in $\mathbb{C}P^2$, *Proceedings of the Japan Academy,* Vol. **67**, Ser. A, No. **10** (1991).

Mohamed Ait Nouh,
Department of Mathematics,
University of California at Riverside
Riverside, CA 92521
Email: maitnouh@math.ucr.edu

Logan Godkin,
Department of Mathematics,
University of California at Riverside
Riverside, CA 92521
lgod@student.ucr.edu