A new invariant of knots via characteristic twisting

Mohamed AIT NOUH

Abstract

We introduce the notion of characteristic twisting of knots in S^3, motivated by characteristic classes. We also define a new invariant of knots corresponding to the minimal number of characteristic twisting disks of a knot k in S^3. The proofs are inspired from Kirby Calculus [14] and some old 4-dimensional techniques.

1. Introduction

Throughout this paper, we work in the smooth category. All orientable manifolds will be assumed to be oriented unless otherwise stated. In particular, all knots are oriented. Let M^4 be a closed 4-manifold and K a knot in $\partial(M^4 - intB^4) \cong S^3$, where B^4 is an embedded 4-ball in M^4. If K bounds a properly embedded 2-disk in $M^4 - intB^4$, then K is a slice knot in M^4.
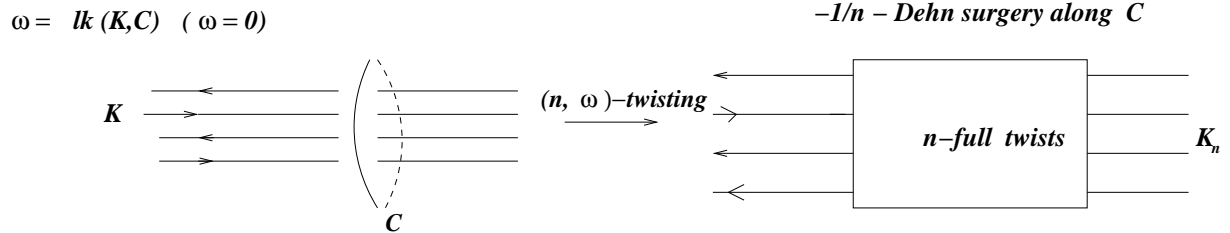

Figure 1:

Let K be a knot in the 3-sphere S^3, and D^2 a disk intersecting K in its interior. Let n be an integer. A $-\frac{1}{n}$-Dehn surgery along ∂D^2 changes K into a new knot K' in S^3. Let $\omega = \text{lk}(\partial D^2, L)$. We say that K' is obtained from K by (n, ω)-twisting (or simply twisting). Then we write $K \overset{(n,\omega)}{\to} K_n$, or $K \overset{(n,\omega)}{\to} K(n, \omega)$. We say that K_n is n-twisted provided that K is the unknot (see Figure 1).

Y. Mathieu asked the following questions in [16]:

Is every knot in S^3 twisted ? If yes what is the minimal number of twisting disks ?

Y. Ohyama [23] showed that any knot k in S^3 can be untied by at most two single twistings. Using 4-dimensional techniques, K. Miyazaki and A. Yasuhara [17] gave an infinite family of prime knots which are non-twisted. We proved that $T(5,8)$ is the smallest non-twistedtorus knot (see the author's Ph. D. thesis [3]), and in a joint work with A. Yasuhara [4], we proved that $T(p, p+4)$ is non-twisted for any $p \geq 7$.

Recall that a homology class $\xi \in H_2(M^4, \mathbb{Z})$ is said to be characteristic provided that $\xi.x \equiv x.x$ for any $x \in H_2(M^4, \mathbb{Z})$ (see [19]).From now on, $\{\bar{\gamma}_1, ..., \bar{\gamma}_m\}$ (resp. $\{\gamma_1, ..., \gamma_m\}$; resp. $\{\alpha, \beta\}$) denote the standard generators of $H_2(m\overline{\mathbb{C}P^2} - intN(B^4), S^3, \mathbb{Z})$ (resp. $H_2(m\mathbb{C}P^2 - intN(B^4), S^3, \mathbb{Z})$; resp. $H_2(S^2 \times S^2 - intN(B^4), S^3, \mathbb{Z})$) with $\alpha^2 = \beta^2 = 0$, $\alpha.\beta = 1$, $\gamma^2 = 1$, $\bar{\gamma}^2 = -1$.

EXAMPLE 1.1. It is easy to check that the homology class $\omega\gamma \in H_2(\mathbb{C}P^2, \mathbb{Z})$ is characteristic if and only if ω is odd. Note also that $a\alpha + b\beta \in H_2(S^2\times, \mathbb{Z})$ is characteristic if and only if a and b are both even. Using Kirby calculus [14], K. Miyazaki and A. Yasuhara [17] showed that if a knot k in S^3 is obtained by (n, ω)-twisting from the unknot, then k bounds a disk Δ, called the twisting disk of k such that:

$$[\Delta] = \begin{cases} \omega(\bar{\gamma}_1 + ... + \bar{\gamma}_n) \in H_2(n\overline{\mathbb{C}P^2} - intB^4, S^3, \mathbb{Z}) & if \quad n > 0 \\ \\ \omega(\gamma_1 + ... + \gamma_n) \in H_2(\mid n \mid \mathbb{C}P^2 - intB^4, S^3, \mathbb{Z}) & if \quad n < 0 \end{cases}$$

Note that $[\Delta]$ is a characteristic class in either cases if and only if ω is odd. Note that the homology class of this disk is given by the number of twisting and the linking number of the twisting disk. However, it is not always characteristic. In the following examples, U denotes the unknot.

EXAMPLE 1.2. We can check that $U \xrightarrow{(-1,2)} 4_1$. Therefore, there exist a smooth disk $(\Delta, \partial\Delta) \subset (\mathbb{C}P^2 - intB^4, S^3)$ such that:

$$\partial\Delta = 4_1 \quad and \quad [\Delta] = 2\gamma \in H_2(\mathbb{C}P^2 - intB^4, S^3, \mathbb{Z}).$$

Note that $[\Delta] = 2\gamma \in H_2(\mathbb{C}P^2 - intB^4, S^3, \mathbb{Z})$ is not characteristic (see [17]).

EXAMPLE 1.3. Note that $U \xrightarrow{(2,2)} 3_1$. Then,there exists $(\Delta, \partial\Delta) \subset (S^2 \times S^2 - intB^4, S^3)$ such that $\partial\Delta = 3_1$ and $[\Delta] = -2\alpha + 2\beta \in H_2(S^2 \times S^2 - intB^4, S^3, \mathbb{Z})$, which is characstic (ref. K. Miyazaki and A. Yasuhara [17], T. Cochran and R.E. Gompf [5]).

EXAMPLE 1.4. $U \cong T(2, -1) \xrightarrow{(2,2)} T(2, 3) \cong T(3, 2) \xrightarrow{(1,3)} T(3, 5)$, then there exist a properly embedded characteristic disk $\Delta \subset S^2 \times S^2 \# \overline{\mathbb{C}P^2} - intB^4$ such that:

$$\partial\Delta = T(3, 5) \quad and \quad [\Delta] = -2\alpha + 2\beta + 3\overline{\gamma} \in H_2(S^2 \times S^2 \# \overline{\mathbb{C}P^2} - intB^4, S^3, \mathbb{Z}).$$

By Kirby Calculus [14], we can prove that any knot in S^3 obtained from a smooth slice knot by a finite series of twisting operations bounds a smooth properly embedded disk Δ in a punctured standard 4-manifold i.e. of the form

$$X^4 = p\mathbb{C}P^2 \# q\overline{\mathbb{C}P^2} \# rS^2 \times S^2 - intB^4 \quad (p, q, r \geq 0).$$

The disk Δ is called the *twisting disk*, and the twisting is called characteristic if the second homology class $[(\Delta, \partial\Delta)] \in H_2(X^4, S^3, \mathbb{Z})$ is characteristic.

The above remarks motivate the following definitions:

DEFINITION 1.1. *Let K be a smooth slice knot in B^4, and k a knot in S^3 such that there exist a positive integer m and a family of Dehn disks $D_1, D_2, ..., D_m$, along which, we perform successively a series of (n_i, ω_i)-twisting:*

$$K \xrightarrow{(n_1, \omega_1)} K(n_1, \omega_1)... \xrightarrow{(n_m, \omega_m)} K((n_1, \omega_1), ..., (n_m, \omega_m)) = k.$$

The characteristic invariant of the isotopy class of k, denoted by $ch_n(k)$, is the minimum over all such integers m among all diagrams of k with the following conditions:

(1) *If ω_i is even for some $i \in \{1, ..., m\}$, then n_i is even.*

(2) *For any $i \in \{1, 2, ...m\}$ we have $\mid \omega_i \mid \leq n$.*

(3) *There exist at least $i \in \{1, 2, ...m\}$ such that $\mid \omega_i \mid = n$.*

(4) *$n_i \neq 0$ for any $i \in \{1, 2, \cdots, m\}$.*

$ch_n(k) = 0$ *if either the four conditions above are not simultaneousely met; for any $\{(n_i, \omega_i)\}_{i=1}^{i=n}$, or k is a smoothly slice knot.*

DEFINITION 1.2. *The characteristic twisting power series expansion of a knot k in S^3 is the Laurent series:*

$$Ch(k) = \sum_{n=0}^{\infty} ch_n(k)X^n$$

DEFINITION 1.3. *The characteristic twisting polynomial of degree $N \geq 0$ of a knot k in S^3 is a Laurent polynomial in a variable X:*

$$Ch_N(k) = \sum_{n=0}^{n=N} ch_n(k)X^n$$

DEFINITION 1.4.
$$Ch_{min}(k) = min\{ch_n(k) > 0\}_{n=0}^{\infty}$$
$$Ch_{max}(k) = max\{ch_n(k)\}_{n=0}^{\infty}$$

REMARK 1.1. Y. Ohyama [23] used Suzuki's diagram [26] to show that any knot k in S^3 can be untied by at most two twistings. More Precisely,$K \xrightarrow{(1,\omega)} K(1,\omega) \xrightarrow{(1,\omega')} k$, which answers Mathieu's question. However, this twisting is not characteristic.

QUESTION 1.1. *Does any knot k in S^3 have a characteristic twisting ? if yes, what is the minimal number of characteristic twisting disks $Ch_{min}(k)$?*

Ch_N does not detect neither amphicheirality nor invertibility i.e. $Ch_N(k) = Ch_N(k^*)(= Ch_N(-k))$, where $-k$ (resp. k^*) is the inverse (resp. the mirror-image knot) of k. We hit the following conjecture:

CONJECTURE 1.1. *Ch is a concordance invariant.*

To answer partially QUESION 1.1, recall that the Δ-move is an unknotting operation introduced by H. Murakami [21] in 1985 (see Figure 2). We denote by $u^{\Delta}(k)$ the minimal number of times this unknotting operation needs to be used to transform a diagram of k into a diagram of the unknot U, where the minimum is taken over all diagrams of k and U.
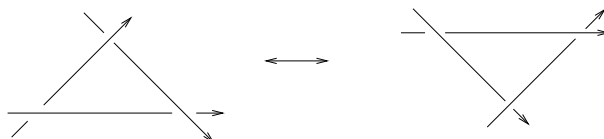

Figure 2:

Theorem 1.1. $1 \leq ch_2(k) \leq 3u^{\Delta}(k)$.

The following Corollary is immediate from Theorem 1.1, answers partially QUESION 1.1.

Corrolary 1.1. $Ch_{min}(k) \leq 3u^{\Delta}(k)$.

Theorem 1.2.

(1) $Ch_3(3_1) = X^2 + X^3$.
(2) $Ch_3(4_1) = 2X^2 + 2X^3$.

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	ch_0	ch_1	ch_2	ch_3	ch_4
Unknot U	*0*	*0*	*0*	*0*	*0*
3_1	*0*	*0*	*1*	*1*	*2*
4_1	*0*	*0*	*2*	*2*	
5_1	*0*	*0*	*1*	*1*	
5_2	*1*	*A*	*X*	*X*	
6_1	*0*	*0*	*0*	*0*	*0*

A=1 or 2
X= 2 or 3

Figure 3:

The table in Figure 3 shows that Ch_4 distinguishes knots up to five crossings. There is no known example of a knot k in S^3 such that $ch_n(k) = 3$ for some $n \geq 0$.

The biggest problem in knot theory is to find a complete invariant that distinguishes between knots. All known polynomial invariants of knots fail to be complete e.g. it is well-known that mutants have the same polyomial invariants of knots. Sometimes, the characteristic twisting of knots can be used to distinguish between knots, where classical invariants fail. Note that the Alexander polynomial does not distinguish between the reef and the granny, whereas the HOMFLY-PT (see [6], [25]) and the Jones polynomial [10] don't distinguish between 5_1 and 10_{132}. We use the characteristic twisting invariant to distinguish between these knots.

Theorem 1.3.

(1) $ch_3(3_1 \# \bar{3}_1) = 0$ and $ch_3(3_1 \# 3_1) = 2$.

(2) $ch_3(5_1) = 1$ and $ch_3(10_{132}) = 2$.

2. Preliminaries

In the following, $b_2^+(M^4)$ (resp. $b_2^-(M^4)$) is the rank of the positive (resp. negative) part of the intersection form of M^4. Let $\sigma(M^4)$ denotes the signature of M^4, and

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ξ^2 the self-intersection of a class ξ in $H_2(M^4, \mathbb{Z})$. For a knot k in S^3, we denote by $\sigma_d(k)$ (resp. $\sigma(k)$) the Tristram d-signature (resp. the signature) of k (A. G. Tristram [27]). To prove the statements, we need the following theorems:

The proof of the following theorem is based on Kirby Calculus [14], and remains true if we substitute the unknot K by any smooth slice knot in B^4.

Theorem 2.1. (Yamamoto -Yasuhara - Weintraub) Let K be a trivial knot in S^3 and $K(n, \omega)$ be the knot obtained from K by (n, ω)-twisting, and $K(n, \omega)(m, \omega')$ is the knot obtained from $K(n, \omega)$ by (m, ω')-twisting: $K \xrightarrow{(n,\omega)} K(n,\omega) \xrightarrow{(m,\omega')} K(n,\omega)(m,\omega')$ If n and ω are even, then there exist a disk D, properly embedded in $S^2 \times S^2 \#(-m)\mathbb{C}P^2 - intN(B^4)$ such that in $H_2(M^4, \partial M^4, \mathbb{Z})$ we have:

$$[D] = \begin{cases} -\omega\alpha + \frac{n\omega}{2}\beta + \omega'(\bar{\gamma}_1 + \bar{\gamma}_2 + ... + \bar{\gamma}_m) & if \quad n > 0, m > 0 \\ \\ \omega\alpha + \frac{n\omega}{2}\beta + \omega'(\gamma_1 + \gamma_2 + ... + \gamma_m) & if \quad n < 0, m < 0 \end{cases}$$

There are several restrictions on embedding of surface of a given genus in a 4-manifolds:

Theorem 2.2. (K. Kikuchi [13]) Let M^4 be a closed, oriented and smooth 4-manifold such that: (1) $H_1(M^4)$ has no 2-torsion; and (2) $b_2^{\pm 1} \leq 3$.

If ξ is a characteristic class of $H_2(M^4, \mathbb{Z})$ represented by an embedded 2-sphere in M^4, then:
$$\xi^2 = \sigma(M^4)$$

P. Gilmer and O. Ya Viro proved independently the following theorem:

Theorem 2.3. (P. Gilmer and O. Ya Viro [7], [28]) Let M^4 be an oriented, compact 4-manifold with $\partial M^4 = S^3$, and K a knot in ∂M^4. Suppose K bounds a surface of genus g in M^4 representing an element ξ in $H_2(M^4, \partial M^4)$. Assume that ξ is divisible by a prime number d.

(1) If d is odd, then we have: $\mid \dfrac{d^2 - 1}{2d}.\xi^2 - \sigma(M^4) - \sigma_d(k) \mid \leq dim H_2(M^4; \mathbb{Z}_d) + 2g$.

(2) If d is even, then we have: $\mid \dfrac{\xi^2}{2} - \sigma(M^4) - \sigma(k) \mid \leq dim H_2(M^4; \mathbb{Z}_2) + 2g$.

Theorem 2.4. (M. Kervaire and J. Milnor [12]) Let M^4 be a smooth closed oriented simply connected 4-manifold. Suppose Σ is an embedded sphere in X of genus g so that $[\Sigma]$ is characteristic. Then

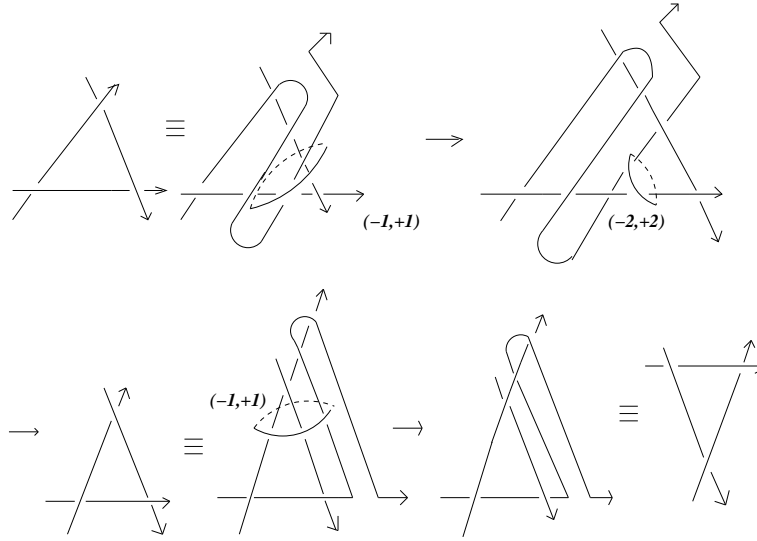$$[\Sigma]^2 \equiv \sigma(M^4) \quad (mod.16)$$

Figure 4:

3. Proof of statements

Proof of Theorem 1.1.

- Figure 4 proves that $ch_2(k) \leq 3u^{\Delta}(k)$.

To prove Theorem 1.1, we need the following lemma:

Lemma 3.1 If a knot k in S^3 is obtained by (n, ω)-twisting from the unknot K along C, then the mirror-image knot k^* is obtained by $(-n, \omega)$-twisting from K, and the inverse knot $-k$ is obtained by $(n, -\omega)$-twisting from K.

Proof. Assume that k is a knot in S^3 obtained from the unknot K by (n, ω)-twisting along the trivial knot $C = \partial D$, where D is the Dehn disk of k. Denote by (m_C, l_C) (resp. (m_{C^*}, l_{C^*})) a paire of preferred meridian-longitude of C (resp. C^*). Then performing a $(-\frac{1}{n})$- Dehn surgery along C corresponds to a homeomorphism $\phi_n : V \mapsto N(C)$ such that $\phi_n(m_V) = m_C l_C^{-n}$. There exist a homeomorphism

$$\Phi : \quad E = S^3 - intN(U \cup C) \quad \mapsto \quad \Phi(E) = S^3 - intN(U \cup C)^* = S^3 - intN(U^* \cup C^*)$$

$$m_C \qquad \mapsto \qquad \Phi(m_C) = m_{C^*}^{-1}$$

$$l_C \qquad \mapsto \qquad \Phi(l_C) = l_{C^*}$$

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$$(U \cup C)^* = U^* \cup C^*$$

$$= \Phi(U) \cup \Phi(C)$$

We denote $E(k) = E\cup_{\phi_n} V$. The homeomorphism Φ can be extended to a homeomorphism: $\tilde{\Phi} : E(k) \mapsto \Phi(E)\cup_{\Phi\circ\phi_n} V$ such that $\Phi(V) = N(C^*)$.

$$
\begin{aligned}
\Phi \circ \phi_n(m_V) &= \Phi(\phi_n(m_V)) \\
&= \Phi(m_C l_C^{-n}) \\
&= m_{C^*}
\end{aligned}
$$

Therefore,

$\tilde{\Phi}(m_C l_C^{-n}) = m_{C^*}^{-1} l_{C^*}^{-n}$ and $\tilde{\Phi}(m_k) = m_{k^*}$. Since the linking number $lk(U, C) = lk(U^*, C^*)$ and $E(k^*) = E(U_{-n}^*)$; then k^* is obtained by $(-n, \omega)$-twisting from the unknot U^*.

To prove that $-k$ is obtained by $(n, -\omega)$-twisting along the unknot, we consider the homeomorphism,

$$\Psi : \quad S^3 - intN(U \cup C) \quad \mapsto \quad S^3 - int(-(U \cup C))$$

$$m_C \quad \mapsto \quad \Psi(m_C) = m_C^{-1}$$

$$l_C \quad \mapsto \quad \Psi(l_C) = l_C^{-1}$$

$$-(U \cup C) = (-U) \cup (-C)$$

$$= \Psi(U) \cup \Psi(C)$$

We denote $E(k) = E\cup_{\phi_n} V$. The homeomorphism Ψ can be extended to a homeomorphism: $\tilde{\Psi} : E(k) \mapsto \Psi(E)\cup_{\Psi\circ\phi_n} V$ such that $\Psi(V) = N(C)$.

$$
\begin{aligned}
\Psi \circ \Psi_n(m_V) &= \Psi(\phi_n(m_V)) \\
&= \Psi(m_C l_C^{-n}) \\
&= m_C^{-1} l_C^{-n}
\end{aligned}
$$

Therefore,

$\tilde{\Psi}(m_C l_C^{n}) = m_C^{-1} l_C^{-n}$ and $\tilde{\Psi}(m_k) = m_{-k}$. Therefore, $E(-k) = E((-U)_n)$. Notice that the linking number $lk(U, C) = -lk(-U, C)$. Therefore, $E(-k) = E((-U)_n)$; which implies that $-k$ is obtained by $(n, -\omega)$-twisting from the unknot $-U$.

Proof of Theorem 1.2. From now on, U_0 denotes a smooth slice knot in B^4. Notice also that whenever we assume that $ch_i(k) = 1$ for some $i \geq 0$, then by Definition 1.1 and Lemma 3.1, we can assume that the number of twisting $n > 0$.

(1) To prove that $Ch_3(3_1) = X^2 + X^3$ is equivalent to proving that

$$(ch_0(3_1), ch_1(3_1), ch_2(3_1), ch_3(3_1)) = (0,0,1,1).$$

(i) To prove that $ch_0(3_1) = 0$, assume for a contradiction that $ch_0(3_1) = m > 0$. By Definition 1.1, there exist a finite series of even integers $n_1, \cdots, n_m$ such that:

$$U_0 \xrightarrow{(n_1,0)} U_0(n_1,0) \longrightarrow \cdots \xrightarrow{(n_m,0)} 3_1$$

Therefore there exist a properly embedded disk $D \subset mS^2 \times S^2 - intB^4$ such that $\partial D = 3_1$ and $[D] = 0 \in H_2(mS^2 \times S^2 - intB^4, S^3, \mathbb{Z})$. Note that $U \xrightarrow{(-1,3)} \bar{3}_1$. Therefore there exist a properly embedded disk $\Delta \subset \mathbb{C}P^2 - intB^4$ such that $\partial \Delta = \bar{3}_1$ and $[\Delta] = 3\gamma \in H_2(\mathbb{C}P^2 - intB^4, S^3, \mathbb{Z})$. Gluing the two characteristic twisting disks along 3_1 yields a characteristic sphere $[D \cup \Delta] = [S]$ whose homology is:

$$[S] = 3\gamma \in H_2(mS^2 \times S^2 \# \mathbb{C}P^2, \mathbb{Z})$$

This would contradict Theorem 2.4, and then $ch_0(3_1) = 0$.

(ii) To prove that $ch_1(3_1) = 0$, assume for a contradiction that $ch_1(3_1) = m$, where $m > 0$. By Definition 1.1, 3_1 would be obtained by a finite series of (n_i, ω_i)-twisting from U_0 ($i = 1, 2, \ldots, m$), where $\omega_i \in \{0, \pm 1\}$ for $i = 1, 2, \ldots, m$; or equivalentely

$$U_0 \xrightarrow{(n_1,0)} U_0(n_1,0) \longrightarrow \cdots \xrightarrow{(n_p,0)} \cdots \xrightarrow{(n_1',\epsilon_1')} \cdots \xrightarrow{(n_q',\epsilon_q')} \cdots \xrightarrow{(n_1'',\epsilon_1'')} \cdots \longrightarrow \cdots \xrightarrow{(n_r'',\epsilon_r'')} 3_1$$

(a) $m = p + q + r$; and

(b) n_i is even for $i = 1, \ldots, p$; and

(c) $n_j' > 0$ and $n_\ell'' < 0$ and $| \epsilon_j' | = | \epsilon_\ell'' | = 1$ for $j = 1, \ldots, q$ and $\ell = 1, \ldots, r$.

Note that $U \xrightarrow{(-1,3)} \bar{3}_1$. By a similar argument as above, gluing the two characteristic twisting disks along 3_1 yields a characteristic sphere whose homology is

$$[S] = 3\gamma + \sum_{i=1}^{i=Q} \bar{\gamma}_i' + \sum_{i=1}^{i=R} \gamma_i'' \in H_2(M^4, \mathbb{Z})$$

Where $Q = \sum_{i=1}^{i=q} n_i'$ and $R = \sum_{i=1}^{i=r} n_i''$ and $M^4 = \mathbb{C}P^2 \# pS^2 \times S^2 \# Q\overline{\mathbb{C}P^2} \# R\mathbb{C}P^2$.

This would contradict Kervaire-Milnor's Theorem 2.4.

(iii) Since $T(2,1) \xrightarrow{(2,2)} 3_1$ and $T(-3,1) \xrightarrow{(1,3)} 3_1$ then $ch_2(3_1) = 1$ and $ch_3(3_1) = 1$
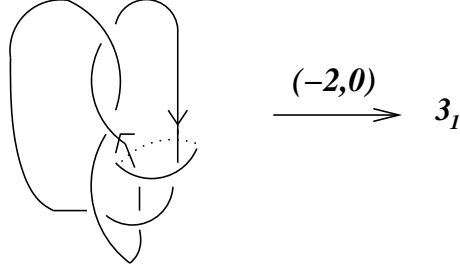
Figure 5:

(2) To prove that $Ch_3(4_1) = 2X^2 + 2X^3$ is equivalent to proving that

$$(ch_0(4_1), ch_1(4_1), ch_2(4_1), ch_3(4_1)) = (0, 0, 2, 2).$$

(i) To prove that $ch_0(4_1) = 0$, assume for a contradiction that $ch_0(4_1) = m$, where $m > 0$. By Definition 1.1, 4_1 would be obtained by a finite series of $(n_i, 0)$-twisting from U_0 and n_i is even $(i = 1, 2, \ldots, m)$ i.e.

$$U_0 \xrightarrow{(n_1,0)} U_0(n_1, 0) \longrightarrow \cdots \xrightarrow{(n_m,0)} 4_1$$

Figure 5 shows that $U \xrightarrow{(2,0)} 3_1 \xrightarrow{(1,3)} 4_1$. Gluing the two twisting disks along 4_1 yields a smooth sphere whose homology is:

$$[S] = 3\bar{\gamma} \in H_2(\overline{\mathbb{C}P^2} \# m S^2 \times S^2, \mathbb{Z})$$

This would contradict Kervaire-Milnor's Theorem 2.4.

(ii) Assume for a contradiction that $ch_1(4_1) = m$, where $m > 0$. By Definition 1.1, 4_1 would be obtained by a finite series of (n_i, ω_i)-twisting from U_0 $(i = 1, 2, \ldots, m)$, where $\omega_i \in \{0, \pm1\}$ for $i = 1, 2, \ldots, m$; or equivalentely

$$U_0 \xrightarrow{(n_1,0)} U_0(n_1, 0) \longrightarrow \cdots \xrightarrow{(n_p,0)} \cdots \xrightarrow{(n_1',\epsilon_1')} \cdots \xrightarrow{(n_q',\epsilon_q')} \cdots \xrightarrow{(n_1'',\epsilon_1'')} \cdots \longrightarrow \cdots \xrightarrow{(n_r'',\epsilon_r'')} 4_1$$

(a) $m = p + q + r$; and

(b) n_i is even for $i = 1, \ldots, p$; and

(c) $n_j' > 0$ and $n_\ell'' < 0$ and $\mid \epsilon_j' \mid = \mid \epsilon_\ell'' \mid = 1$ for $j = 1, \ldots, q$ and $\ell = 1, \ldots, r$.

Note that $U \xrightarrow{(1,3)} 3_1 \xrightarrow{(2,0)} 4_1$ as depicted in Figure 5. By a similar argument as above, gluing the two characteristic twisting disks along 4_1 yields a characteristic sphere whose homology is

$$[S] = 3\bar{\gamma} + \sum_{i=1}^{i=Q} \bar{\gamma}_i' + \sum_{i=1}^{i=R} \gamma_i'' \in H_2(M^4, \mathbb{Z})$$

Where $Q = \sum\limits_{i=1}^{i=q} n_i'$ and $R = \sum\limits_{i=1}^{i=r} n_i''$ and $M^4 = \overline{\mathbb{CP}^2}\#(p+1)S^2 \times S^2 \# Q\mathbb{CP}^2 \# R\mathbb{CP}^2$.
This would contradict Kervaire-Milnor's Theorem 2.4.

(iii) To prove that $ch_2(4_1) = 2$, note that Figure 5 shows that $U \xrightarrow{(2,2)} 3_1 \xrightarrow{(2,0)} 4_1$ and then $ch_2(4_1) \leq 2$. The assumption $ch_2(4_1) = 1$ would contradict Gilmer-Viro's Theorem 2.1.(1); and therefore $ch_2(4_1) = 2$.

(iv) To prove that $ch_3(4_1) = 2$, Figure 5 also proves that $U \xrightarrow{(1,3)} 3_1 \xrightarrow{(2,0)} 4_1$. The assumption $ch_3(4_1) = 1$ would contradict Theorem 2.1.(2) with $d = 3$.

Proof on Theorem 1.3.

(1) Since $3_1\#\bar{3}_1$ is smoothly slice, then by Definition 1.1, $ch_3(3_1\#\bar{3}_1) = 0$. K. Miyazaki and A. Yasuhara proved in [17] that $3_1\#3_1$ is non-twisted. It is easy to check that $U \xrightarrow{(1,3)} 3_1 \xrightarrow{(1,3)} 3_1\#3_1$,and therefore $ch_3(3_1\#3_1) = 2$.
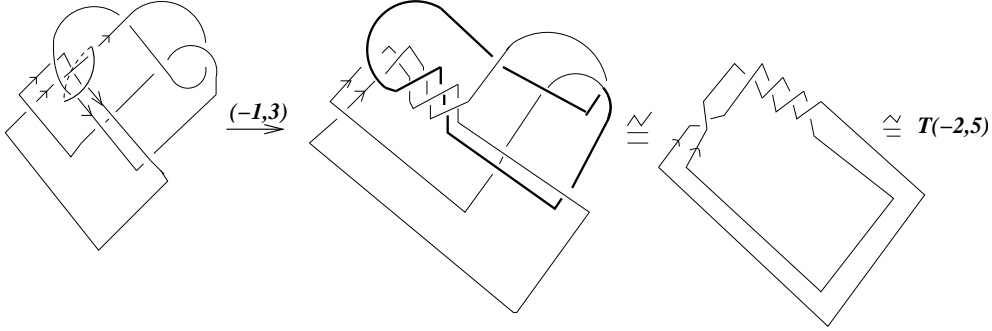

Figure 6:

(2) $U \xrightarrow{(1,3)} T(2,5)$ (Goda-Hayashi-Song [8]), then $ch_3(T(2,5)) = 1$. To show that $ch_3(10_{132}) = 2$, notice that $10_{132} \xrightarrow{(-1,3)} T(-2,5) \xrightarrow{(2,2)} U$ (see Figure 6), or equivalently, by Lemma 3.1, $U \xrightarrow{(2,2)} T(-2,5) \xrightarrow{(-1,3)} 10^*_{132}$. Then $ch_3(10_{132}) \leq 2$. Assume for a contradiction that $ch_3(10_{132}) = 1$, then $U \xrightarrow{(n,3)} 10_{132}$ where $n > 0$. Therefore there exist a properly embedded disk $D \subset n\overline{\mathbb{CP}^2} - intB^4$ such that $\partial D = 10_{132}$ and $[D] = 3\sum\limits_{i=1}^{i=n} \bar{\gamma}_i \in H_2(n\overline{\mathbb{CP}^2} - intB^4, S^3, \mathbb{Z})$. Note that $\sigma_d(10_{132}) = 0$ for any prime $d \geq 2$. Indeed, since the argument of the roots of the Alexander polynomial of 10_{132} do not lie in $[\frac{2\pi}{3}, \pi]$, then the Tristram d-signature of 10_{132} do not depend on d (see K. Miyazaki and A. Yasuhara [20]). Since $\sigma(10_{132}) = 0$, then $\sigma_d(10_{132}) = 0$ for any prime $d \geq 2$. The assumption $ch_3(10_{132}) = 1$ would contradict Theorem 2.1.(2) with $d = 3$.

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Department of Mathematics,
University of California, Riverside
900 University Drive, Riverside, CA 93021
e-mail: maitnouh@math.ucr.edu