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Obtaining graph knots by twisting unknots

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Abstract

Let K be a knot in the 3-sphere S^3 and D a disk in S^3 meeting K transversely more than once in the interior. For nontriviality we assume that $|D \cap K| \geqslant 2$ over all isotopies of K in $S^3 - \partial D$. Let $K_{D,n}$ $(\subset S^3)$ be a knot obtained from K by n twisting along the disk D. We prove that if K is a trivial knot and $K_{D,n}$ is a graph knot, then $|n| \leqslant 1$ or K and D form a special pair which we call an "exceptional pair". As a corollary, if (K, D) is not an exceptional pair, then by twisting unknot K more than once (in the positive or the negative direction) along the disk D, we always obtain a knot with positive Gromov volume. We will also show that there are infinitely many graph knots each of which is obtained from a trivial knot by twisting, but its companion knot cannot be obtained in such a manner.

MSC: 57M25

Keywords: Twisting; Gromov volume; Graph knots

1. Introduction

Let K be a knot in the 3-sphere S^3 and D a disk in S^3 meeting K transversely more than once in the interior. We assume that $|D \cap K|$ is minimal and greater than one over all isotopies of K in $S^3 - \partial D$. We call such a disk D a *twisting disk* for K. Let $K_{D,n}(\subset S^3)$

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be a knot obtained from K by n twisting along the disk D, in other words, $-\frac{1}{n}$-surgery on the trivial knot ∂D.

A knot in S^3 is called a *graph knot* if its exterior is a graph manifold, i.e., there is a family of tori which decompose the exterior into Seifert fiber spaces.

Let us introduce some typical twistings which convert unknots into graph knots.

Definition (*Exceptional pair*). Let K^0 be a trivial knot intersecting a disk D exactly once; $K \cup \partial D$ be a Hopf link in S^3. We define K^i to be an (ε_i, q_i)-cable of K^{i-1} $(1 \leqslant i \leqslant m)$, i.e., K^i is an essential, simple closed curve on the boundary of a small tubular neighborhood of K^{i-1} wrapping ε_i (respectively q_i) times in meridional (respectively longitudinal) direction, where $|\varepsilon_i| = 1$ and $q_i \geqslant 2$. Then K^m is a trivial knot in S^3 and $K^m_{D,n}$ is an iterated torus knot for any integers m and n; in particular, $K^1_{D,n}$ is an $(\varepsilon_1 + nq_1, q_1)$-torus knot $T(\varepsilon_1 + nq_1, q_1)$ and if further $q_1 = 2$ then $K^1_{D,-\varepsilon_1}$ is a trivial knot, see Fig. 1 in which $m = 1$. A pair (K, D) is called an *exceptional pair* of type $(\varepsilon_1, q_1; \ldots; \varepsilon_m, q_m)$ if the link $K \cup \partial D$ is isotopic to a link $K^m \cup \partial D$ for some integer m.

In this paper we will prove:

Theorem 1.1. *Suppose that K is a trivial knot and D a twisting disk for K. If a knot $K_{D,n}$ is a graph knot, then $|n| \leqslant 1$ or (K, D) is an exceptional pair.*

Here are some examples of non-exceptional pairs (K, D) such that $K_{D,1}$ is a graph knot.

Example 1. In Fig. 2, $K_{D,1}$ is a trefoil knot. In [5], [32, p. 2293], we find other examples of non-exceptional pairs (K, D) such that $K_{D,1}$ or $K_{D,-1}$ is a torus knot.

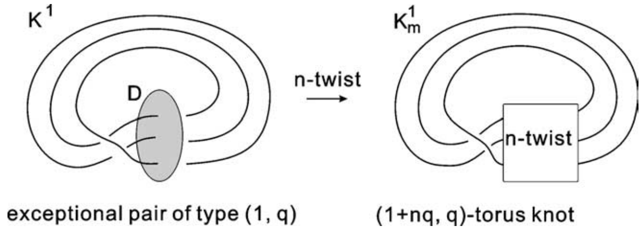

Fig. 1.

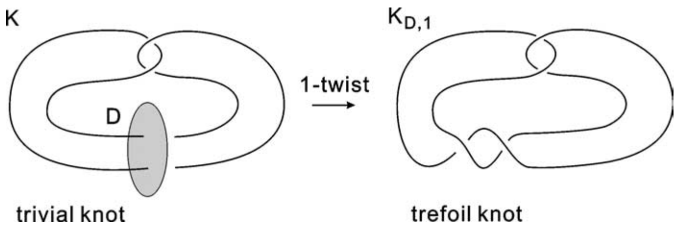

Fig. 2.

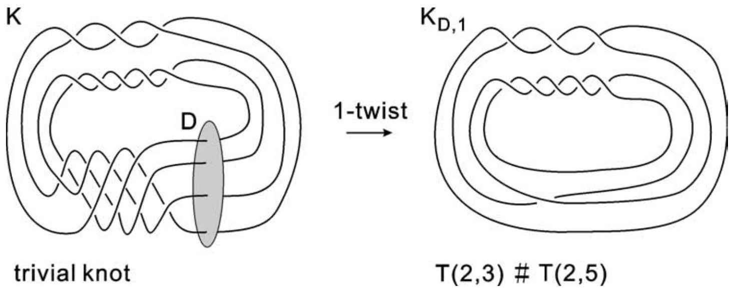

Fig. 3.

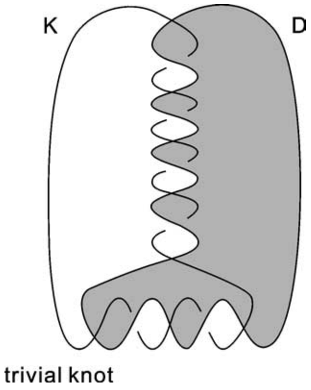

Fig. 4.

Example 2. In Fig. 3, $K_{D,1}$ is a connected sum of two torus knots [25].

Example 3 [5,32]. For the pair (K, D) in Fig. 4, $K_{D,1}$ is a $(23, 2)$-cable of a $(4, 3)$-torus knot. By [21] the link $K \cup \partial D$ is hyperbolic, in particular, (K, D) is a non-exceptional pair.

Once we have a non-exceptional pair (K, D) such that $K_{D,1}$ is a graph knot, we can obtain another pair (K', D) by taking some cables of K.

Example 4. Let (K, D) be a pair given in Example 1, 2 or 3. Applying a construction of exceptional pair to the pair (K, D) instead of the Hopf link (K^0, D), we can obtain a non-exceptional pair (K', D) so that $K'_{D,1}$ is a graph knot which is an iterated cable of $K_{D,1}$.

We then apply Theorem 1.1 to a study of Gromov volumes $\|K_{D,n}\|$. For the definition of Gromov volumes, see [13], [29, Section 6], [28]. It is convenient for us to recall some properties of Gromov volumes.

- Let K be a hyperbolic knot, i.e., its complement admits a complete hyperbolic metric. Then $\|K\| = \frac{\mathrm{Vol}(S^3 - K)}{v_3}$, where $\mathrm{Vol}(S^3 - K)$ is the volume of $S^3 - K$ and v_3 is the volume of the regular ideal simplex. More generally, if P is a hyperbolic manifold with toral boundary, then $\|P\| = \frac{\mathrm{Vol}(P)}{v_3}$ [29].

- Let K be a torus knot, i.e., its exterior is a Seifert fiber space, then $\|K\| = 0$. More generally, if P is a Seifert fiber space, then $\|P\| = 0$ [29].
- Let K be a satellite knot with a family of essential tori $\mathcal{T}$. Let P_i $(1 \leqslant i \leqslant n)$ be the closure of a component of $E(K) - \mathcal{T}$. Then $\|K\| = \sum_{i=1}^{n} \|P_i\|$ [28].

It follows that a knot is a graph knot if and only if its Gromov volume vanishes. Thus we have:

Corollary 1.2 (Gromov volumes). *Let K be a trivial knot and (K, D) a non-exceptional pair. If $|n| > 1$, then $\|K_{D,n}\| > 0$.*

If (K, D) is an exceptional pair, then $K_{D,n}$ is an iterated torus knot and $\|K_{D,n}\| = 0$ for any integer n.

Remark. For any $r \in \mathbb{R}$, we can take a twisting disk D for the trivial knot K so that $\|K_{D,1}\| > r$, see [19, Proposition 3.3].

In Example 2 above, the graph knot $T_{2,3} \sharp T_{2,5}$ can be obtained from a trivial knot by twisting and its companion knots $T_{2,3}$ and $T_{2,5}$ can be also obtained from a trivial knot by twisting. Furthermore, in Example 4 every companion knot of $K'_{D,1}$ is also obtained from a trivial knot by twisting. So it is natural to ask: if a satellite knot (not necessarily a graph knot) k can be obtained from a trivial knot by twisting, then can every companion knot be obtained in such a manner?

The next proposition answers this question in the negative.

Proposition 1.3. *There exists an infinite family of composite graph knots each of which can be obtained by twisting a trivial knot, but its companion knot is not obtained in such a manner.*

Proof. Denote the (p, q)-torus knot by $T(p, q)$, where $0 < p < q$, p and q are coprime integers. The knot $k = T(p, p + 4) \sharp T(-p, 2p + 4)$ can be obtained from a trivial knot by twisting, see [6, Appendix B.2]. However, the companion knot $T(p, p + 4)$ cannot be obtained from a trivial knot by twisting, see [1]. □

2. Satellite diagrams

To simplify descriptions, here we recall *satellite diagrams* [22]. Let k be a nontrivial knot in S^3. Let $\mathcal{T}$ be a (possibly empty) set of essential tori in $E(k) = S^3 - \mathrm{int}\, N(k)$ which gives the torus decomposition of $E(k)$ in the sense of Jaco and Shalen [15] and Johannson [17]. The closure of each component of $E(k) - \bigcup \mathcal{T}$, which is referred to as a *decomposing piece*, is hyperbolic or Seifert fibered; moreover, a Seifert fibered piece is either a torus knot space, a cable space, or a composing space [15, Lemma VI.3.4.]. A satellite diagram,

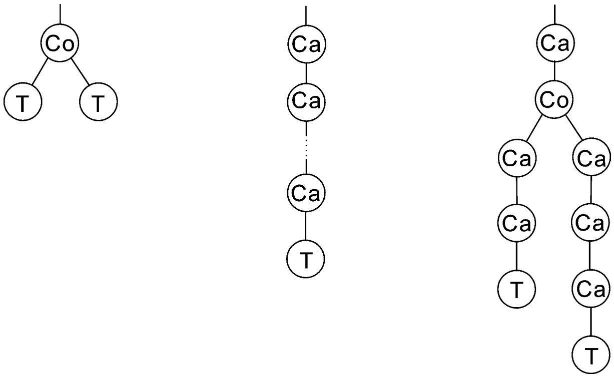

Fig. 5.

D say, for k is a tree with labelled vertices and one open edge defined as follows. Each vertex of D corresponds to a decomposing piece, each edge of D corresponds to a torus in $\mathcal{T} \cup \partial E(k)$, each vertex is labelled $T, Ca, Co,$ or H according as the decomposing piece is a torus knot space, a cable space, a composing space, or a hyperbolic space, respectively. Note that an edge for a torus in $\mathcal{T}$ connects two vertices, but the edge for $\partial E(k)$ has one end open. If k is *simple* (i.e., $\mathcal{T} = \emptyset$), then the satellite diagram consists of a single vertex with one open edge. For example, the satellite diagram for a connected sum of two torus knots, an iterated torus knot and a cable of a connected sum of two iterated torus knots are given in Fig. 5. For a given knot k, since the torus decomposition of $E(k)$ is unique up to isotopy, the satellite diagram for k is uniquely determined.

A knot k is a graph knot, equivalently the Gromov volume of k is vanishes, if and only if each label appeared at vertices of the satellite diagram is T, *Ca* or *Co*.

The vertex corresponding to a decomposing piece which contains $\partial E(k)$ is called the *innermost vertex*. Note that if the innermost vertex is T (respectively *Ca* or *Co*), then k is a torus knot (respectively a cable knot or a composite knot).

3. Planar surfaces in graph knot exteriors

Let k be a cable knot (which may be a torus knot). Then there is a (possibly unknotted) solid torus V in S^3 such that k is lying on the boundary ∂V and k wraps more than once in longitudinal direction. Then an annulus $A = \partial V - \operatorname{int} N(k)$ is essential, meaning incompressible and boundary-incompressible, in $S^3 - \operatorname{int} N(k)$. We call such an annulus A a *cabling annulus* of k. It is known that every essential planar surface in a torus knot space is isotopic to a cabling annulus [30]. The goal in this section is to prove the analogous result for graph knots.

Let k be a (nontrivial) prime graph knot. Then the innermost vertex of the satellite diagram of k has a label *Ca* (i.e., a decomposing piece P_1 which contains $\partial E(k)$ is a cable space) and k is a cable knot.

A *slope* on $\partial N(k)$ is the isotopy class of a simple closed curve on $\partial N(k)$. Let F be an essential planar surface. Then all the boundary components of F are essential on $\partial N(k)$ and have the same slope, the *boundary slope* of F.

Proposition 3.1. *Let k be a* (*nontrivial*) *prime graph knot in S^3. Every essential planar surface in $E(k)$ whose boundary slope is not $\frac{1}{0}$ is isotopic to a cabling annulus.*

Proof. If k is a torus knot, then the result follows from [30]. We hereafter assume that k is a satellite knot.

Let F be an essential planar surface in $E(k)$ whose boundary slope is not a meridian. We begin by observing that F is separating. Assume for a contradiction that F does not separate $E(k)$. Then each component of ∂F represents a longitudinal slope of k. Thus 0-surgery on k produces a manifold which contains a non-separating 2-sphere. This implies that k is a trivial knot [4, Corollary 8.3], contradicting the assumption.

Let $\mathcal{T}$ be a (non-empty) family of tori which defines a torus decomposition of $E(k)$, i.e., $\mathcal{T}$ decomposes $E(k)$ as $E(k) = \bigcup P_i$. Since k is a graph knot, each P_i is a torus knot space, a cable space or a composing space. Furthermore since k is prime, the piece containing $\partial E(k)$ is a cable space. If P_i is a cable space, then ∂P_i consists of two components; the component which is closer to $\partial E(k)$ is called an *inner boundary component* and the other component is called an *outer boundary component*. Note that each boundary component of P_i bounds a solid torus in S^3 containing k in its interior. We use the slope $\frac{a}{b}$ in the preferred meridian-longitude co-ordinate determined by the solid torus; it will be assumed that $b \geqslant 0$. For a properly embedded surface $F_i \subset P_i$, the *inner* (respectively *outer*) boundary component of F_i is the component lying on the inner (respectively outer) boundary of P_i. Similarly the slope represented by an inner (respectively outer) boundary component of F_i is referred to as the *inner* (respectively *outer*) *boundary slope* of F_i. Note that every component of ∂F is contained in the inner boundary component of P_1.

We isotope F so that F intersects $T_i\, (\in \mathcal{T})$ transversely and $|F \cap (\bigcup_{T_i \in \mathcal{T}} T_i)|$ is minimal. Then each component of $F \cap P_i$ is a properly embedded planar surface in P_i.

Claim 3.2. *Each component of $F \cap P_i$ is an essential surface in P_i.*

Proof. Assume for a contradiction that a component F' of $F \cap P_i$ is compressible in P_i. Then there is a compressing disk $\Delta (\subset P_i)$ for F'. Since F is incompressible in $E(k)$, $\partial \Delta$ bounds a disk D in F. Since $\partial \Delta = \partial D$ is essential in F', $D \cap \partial P_i \neq \emptyset$. Let c be an innermost circle in $D \cap (\bigcup_{T_j \in \mathcal{T}} T_j)$ and $D_c \subset D$ the disk bounded by c. Assume that D_c is contained in a decomposing piece P_j. A boundary-irreducibility and an irreducibility of P_j, we see that D_c is a boundary-parallel disk in P_j. Thus we can remove c by an isotopy. This contradicts the minimality of $|F \cap (\bigcup_{T_i \in \mathcal{T}} T_i)|$. Hence each component of $F \cap P_i$ is incompressible in P_i.

If some component of $F \cap P_i$ is boundary-compressible in P_i, then it should be a boundary-parallel annulus. This contradicts again the minimality of $|F \cap (\bigcup_{T_i \in \mathcal{T}} T_i)|$. □

Let us recall the following.

Lemma 3.3 [8]. *Every incompressible, boundary-incompressible connected planar surface in a (p,q)-cable space is of one of the following types*:

(1) *an annulus with both boundary components inner, of slope pq*;
(2) *an annulus with both boundary components outer, of slope $\frac{p}{q}$*;
(3) *an annulus with one inner boundary component of slope pq, and one outer boundary component of slope $\frac{p}{q}$*;
(4) *a surface with q inner boundary components of slope $\frac{1+kpq}{k}$, and one outer boundary component of slope $\frac{1+kpq}{kq^2}$, for some integer k*;
(5) *a surface with one inner boundary component of slope $\frac{\ell q^2}{m}$, and q outer boundary components of slope $\frac{\ell}{m}$, for some integers ℓ and m such that $\ell q = 1 + mp$.*

A (p,q)-cable space ($q \geqslant 2$) has a unique Seifert fibration up to isotopy. A surface in the cable space is isotopic to a vertical (a union of fibers) annulus if and only if it is of type (1), (2) or (3), and is isotopic to a horizontal (transverse to fibers) surface if and only if it is of type (4) or (5). An essential annulus in $E(k)$ is a cabling annulus if it is isotopic to an annulus in P_1 with type (1).

We divide the proof into two cases depending on whether the satellite diagram has a vertex with label *Co* or not.

Case (I). The satellite diagram of k has no vertices with label *Co*, i.e., k is an iterated torus knot.

Then we put the decomposing pieces $P_1, P_2, \dots, P_m$ so that P_i is the ith closest piece from k; P_i $(1 \leqslant i \leqslant m-1)$ is a (p_i, q_i)-cable space and P_m is a (p_m, q_m)-torus knot space. Let n be the largest number such that $F \cap P_n \neq \emptyset$. Then P_n is a cable space (respectively a torus knot space) if $n < m$ (respectively $n = m$).

Claim 3.4. *Each component of $F \cap P_n$ is a vertical annulus.*

Proof. Let F' be a component of $F \cap P_n$. First suppose that P_n is a torus knot space. Since F' is an essential planar surface in P_n, F' is isotopic to a vertical annulus [30]. Next suppose that P_n is a cable space. Then by the choice of P_n, $\partial F'$ is contained in the inner boundary component of P_n. From Lemma 3.3 we see that F' is isotopic to a vertical annulus. □

To prove Proposition 3.1, it is sufficient to show that $n = 1$. In fact, once we establish that $n = 1$, then the planar surface $F \subset E(k)$ (which was isotoped so that $|F \cap (\bigcup_{T_i \in \mathcal{T}} T_i)|$ is minimal) is contained in P_1 with only inner boundary components, and hence it is a cabling annulus as desired. Let us assume for a contradiction that $n \geqslant 2$.

By Claim 3.4, $F \cap P_n$ consists of vertical annuli, hence each component F_n of $F \cap P_n$ has the inner boundary slope $p_n q_n$, see Lemma 3.3. On the other hand, $F \cap P_{n-1}$ is isotopic to a horizontal surface, for otherwise, $F \cap P_{n-1}$ is also isotopic to a vertical surface and Seifert fibrations of P_{n-1} and P_n match and hence $P_{n-1} \cup P_n$ is also a Seifert fiber space, a contradiction. Hence each component of $F \cap P_{n-1}$ is of type (4) or (5) in Lemma 3.3. If some component F_{n-1} is of type (4), then the outer boundary

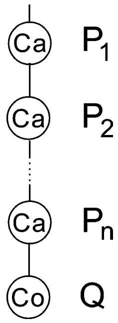

Fig. 6.

slope of F_{n-1} equals $\frac{1+kp_{n-1}q_{n-1}}{kq_{n-1}^2}$ for some integer k, which coincides with the integer p_nq_n. This is impossible because $q_{n-1} \geqslant 2$. Hence each component of $F \cap P_{n-1}$ is of type (5). Let us take a connected component $F_{n,n-1}$ of $F \cap (P_n \cup P_{n-1})$. Then $F_{n,n-1}$ has a form $(\bigcup_{1\leqslant i\leqslant x} F_{n-1}^i) \cup (\bigcup_{1\leqslant j\leqslant y} F_n^j)$, where F_{n-1}^i is a component of $F \cap P_{n-1}$ and F_n^j is that of $F \cap P_n$. The boundary of $F_{n,n-1}$ consists of the inner boundary components of F_{n-1}^i $(1 \leqslant i \leqslant x)$; each F_{n-1}^i has exactly one inner boundary component (see Lemma 3.3), hence $F_{n,n-1}$ is an x-punctured 2-sphere. Since F_n^j is an annulus and F_{n-1}^i is a disk with q_{n-1} holes (see Lemma 3.3), the Euler characteristic $\chi(F_{n,n-1}) = \chi((\bigcup_{1\leqslant i\leqslant x} F_{n-1}^i) \cup (\bigcup_{1\leqslant j\leqslant y} F_n^j))$ is $x(1-q_{n-1})$, which should be $2-x$. However, this is impossible because $q_{n-1} \geqslant 2$. It follows that $n = 1$ and F is a cabling annulus.

Case (II). The satellite diagram of k has a vertex with label *Co*, i.e., there is a composing space in $E(k)$.

Then we can find a sub-tree as in Fig. 6.

Let Q be the closest composing space in $E(k)$ and denote the cable spaces $P_1, P_2, \ldots, P_n$ so that P_i is the ith closest piece from k; Q is the $(n+1)$-closest piece from k. The closure of $E(k) - \bigcup_{i=1}^n P_i$ is the exterior $E(k')$ of a companion knot k' of k.

If $F \cap Q = \emptyset$, then we can reduce case (II) to case (I), thus to complete the proof of Proposition 3.1, we will assume that $F \cap Q \neq \emptyset$ and derive a contradiction.

Let F_n be a component of $F \cap P_n$ such that F_n intersects both inner and outer boundary components of P_n; since $F \cap Q \neq \emptyset$ and F is connected, such a component exists. Let F' be a component of $F \cap E(k')$ such that $\partial F' \cap \partial F_n \neq \emptyset$.

Now we divide into two subcases.

Case (II)-(a). F_n is vertical, i.e., F_n is an annulus with one inner boundary component of slope p_nq_n, and one outer boundary component of slope $\frac{p_n}{q_n}$, see Lemma 3.3. Since F_n has the outer boundary slope $\frac{p_n}{q_n}$, the boundary slope of F' is also $\frac{p_n}{q_n}$. It is easy to observe that F' is an essential planar surface in $E(k')$ (cf. Claim 3.2). On the other hand, since Q is a composing space (homeomorphic to [disk with holes] $\times S^1$), we can find an essential annulus A in $E(k')$ with $\partial A \subset \partial E(k')$ such that its boundary slope is $\frac{1}{0}$. Then [11, Theorem 1.1] asserts that $q_n \leqslant 1$, contradicting the fact that $q_n \geqslant 2$.

Case (II)-(b). F_n is horizontal, i.e., F_n is of type (4) or (5) in Lemma 3.3.

Case (II)-(b)-type (4). First suppose that F_n is of type (4). Then the outer (respectively inner) boundary slope of F_n is $\frac{1+k_n p_n q_n}{k_n q_n^2}$ (respectively $\frac{1+k_n p_n q_n}{k_n}$) for some integer k_n, see Lemma 3.3. Since the outer boundary slope of F_n coincides with the boundary slope of $F' \subset E(k')$, the argument in case (II)-(a) above shows $|k_n q_n^2| \leqslant 1$. Since $q_n \geqslant 2$, we have $k_n = 0$. Thus the inner boundary slope of F_n is $\frac{1+k_n p_n q_n}{k_n} = \frac{1}{0}$.

Let F_{n-1} be a component of $F \cap P_{n-1}$ such that F_{n-1} intersects both inner and outer boundary components of P_{n-1}; since $F \cap P_n \neq \emptyset$ and F is connected, such a component exists.

Claim 3.5. *The inner boundary slope of F_{n-1} is $\frac{1}{0}$.*

Proof. If F_{n-1} is isotopic to a vertical annulus, then F_{n-1} is of type (3) and the outer boundary slope is $\frac{p_{n-1}}{q_{n-1}}$, which coincides with $\frac{1}{0}$ (the inner boundary slope of F_n). This contradicts that $q_{n-1} \geqslant 2$. This then implies that F_{n-1} is of type (4) or (5).

First suppose that F_{n-1} is of type (4). By Lemma 3.3, the outer boundary slope of F_{n-1} is $\frac{1+k_{n-1}p_{n-1}q_{n-1}}{k_{n-1}q_{n-1}^2}$ and the inner boundary slope F_{n-1} is $\frac{1+k_{n-1}p_{n-1}q_{n-1}}{k_{n-1}}$ for some integer k_{n-1}. Recall that the inner boundary slope of F_n which coincides with the outer boundary slope of F_{n-1} is $\frac{1}{0}$. It follows that $k_{n-1} = 0$ (because $q_{n-1} \geqslant 2$) and hence the inner boundary slope $F_{n-1} = \frac{1}{0}$ as required.

Next suppose that F_{n-1} is of type (5). Then again by Lemma 3.3, the outer boundary slope of F_{n-1} is $\frac{\ell_{n-1}}{m_{n-1}}$ and the inner boundary slope of F_{n-1} is $\frac{\ell_{n-1}q_{n-1}^2}{m_{n-1}}$ for some integers ℓ_{n-1}, m_{n-1}. The above argument shows that $m_{n-1} = 0$ and hence the inner boundary slope of F_{n-1} is also $\frac{1}{0}$ as required. □

Applying the argument in Claim 3.5 successively, we can conclude that the inner boundary slope of F_1, which is the boundary slope of F, is $\frac{1}{0}$, contradicting the initial assumption.

Case (II)-(b)-type (5). Let us suppose that F_n is of type (5). Then the outer (respectively inner) boundary slope of F_n is $\frac{\ell_n}{m_n}$ (respectively $\frac{\ell_n q_n^2}{m_n}$) for some integer ℓ_n, m_n (Lemma 3.3). Since the outer boundary slope of F_n coincides with the boundary slope of $F' \subset E(k')$, the argument in case (II)-(a) shows that $m_n = 0, 1$.

Assume that $m_n = 0$. Then the inner boundary slope of F_n is $\frac{\ell_n q_n^2}{m_n} = \frac{1}{0}$. This means that the outer boundary slope of F_{n-1} is $\frac{1}{0}$. Then the identical argument in case (II)-(b)-type (4) shows that the inner boundary slope of F_1, which is the boundary slope of F, is $\frac{1}{0}$, contradicting the initial assumption.

Assume that $m_n = 1$. Then the inner boundary slope of F_n is $\frac{\ell_n q_n^2}{m_n} = \ell_n q_n^2$.

Claim 3.6. *Each component of $F \cap P_1$ is of type* (5) *in the cable space P_1.*

Proof. Take a component F_{n-1} of $F \cap P_{n-1}$ so that F_{n-1} intersects both inner and outer boundary components of P_{n-1}. Since the inner boundary slope of F_n is $\ell_n q_n^2$, the outer boundary slope of F_{n-1} is also the integer $\ell_n q_n^2$. If F_{n-1} is isotopic to a vertical annulus, then the outer boundary slope is $\frac{p_{n-1}}{q_{n-1}}$, which cannot be an integer, because $(p_{n-1}, q_{n-1}) = 1$ and $q_{n-1} \geqslant 2$. If F_{n-1} is of type (4), then the outer boundary slope is $\frac{1+k_{n-1}p_{n-1}q_{n-1}}{k_{n-1}q_{n-1}^2}$, which cannot be an integer, for $q_{n-1} \geqslant 2$. Thus we assume that F_{n-1} is of type (5). Then the outer boundary slope is $\frac{\ell_{n-1}}{m_{n-1}}$, which is an integer only if $m_{n-1} = 1$. This then implies that the inner boundary slope of F_{n-1} equals $\frac{\ell_{n-1}q_{n-1}^2}{m_{n-1}} = \ell_{n-1}q_{n-1}^2$. Repeating this argument, we see that each component F_1 of $F \cap P_1$ is of type (5). This completes the proof. □

Now we will show that the situation in Claim 3.6 cannot happen.

Suppose for a contradiction that $F \cap P_1$ consists of surfaces of type (5), say $F_1^1, \ldots, F_1^x$ each of which is a planar surface with one inner boundary component and q_1 outer boundary components. Write $F \cap (E(k) - \operatorname{int} P_1) = F'_1 \cup \cdots \cup F'_y$, where F'_i is a connected planar surface with t_i boundary components ($i = 1, \ldots, y$).

Claim 3.7. *F'_i $(1 \leqslant i \leqslant y)$ is not a disk, and hence $t_i \geqslant 2$.*

Proof. Assume for a contradiction that F'_i is a disk. Let c be an innermost circle in $F'_i \cap (\bigcup_{T_j \in \mathcal{T}} T_j)$ and $D_c \subset F'_i$ the disk bounded by c. (Possibly $c = \partial F'_i$ and $D_c = F'_i$.) Assume that D_c is contained in a decomposing piece P of $E(k)$. Since P is irreducible and boundary-irreducible, we see that D_c is a boundary-parallel disk in P. Thus we can remove c by an isotopy. This contradicts the minimality of $|F \cap (\bigcup_{T_i \in \mathcal{T}} T_i)|$. □

Note that $F = (\bigcup_{i=1}^x F_1^i) \cup (\bigcup_{i=1}^y F'_i)$ is an x-punctured sphere. Consider Euler characteristic, we have $2 - x = x(1 - q_i) + \sum_{i=1}^y (2 - t_i)$, i.e., $2 = x(2 - q_1) + \sum_{i=1}^y (2 - t_i)$. Since $q_1 \geqslant 2$ and $t_i \geqslant 2$, the right-hand side of the equation is not positive, a contradiction.

It follows that $F \cap Q = \emptyset$ and the proof of Proposition 3.1 is now completed. □

4. Proof of Theorem 1.1 for hyperbolic pairs

Let K be a knot in S^3 and D a twisting disk for K. Set $c = \partial D$. We say that the pair (K, D) is a *hyperbolic pair* if the link $K \cup c$ is hyperbolic, i.e., $S^3 - K \cup c$ is hyperbolic.

The goal in this section is to prove Theorem 1.1 for hyperbolic pairs. It should be mentioned that if (K, D) is a hyperbolic pair and $K_{D,n}$ is a satellite knot, then as a particular case of [12] we can deduce that $n \leqslant 2$.

Proposition 4.1. *Suppose that (K, D) is a hyperbolic pair. If $K_{D,n}$ is a graph knot, then $|n| \leqslant 1$.*

We attempt to follow, verbatim, the proof of [24, Proposition 2.1]. Before proving the proposition, we prepare some notations.

Let K be a knot in a 3-manifold M. The manifold obtained from M by Dehn surgery on a knot K with slope γ is denoted by $M(K; \gamma)$; if $M \cong S^3$, for simplicity we denote

$M(K;\gamma)$ by $(K;\gamma)$. If $M \subset S^3$, then using the preferred meridian-longitude pair of $K \subset S^3$, we parameterize slopes γ of K by $r \in \mathbb{Q} \cup \{\infty\}$, then we also write $(K;r)$ for $(K;\gamma)$. A slope of K is *integral* if a representative of it intersects a meridian of K exactly once. For knots in S^3 integral slopes correspond to integers.

Recall that in our setting, K is a trivial knot and the exterior $E(K) = S^3 - \operatorname{int} N(K)$ is a solid torus containing c in its interior. Let (μ_0, λ_0) be a preferred meridian-longitude pair of K. By performing $-\frac{1}{n}$-surgery on c, we obtain a twisted knot K_n in S^3 as the image of K. Let (μ_n, λ_n) be a preferred meridian-longitude pair of K_n.

The preferred meridian-longitude pairs of K and that of K_n are related as follows (for suitable orientations). We omit the proof here.

Claim 4.2. *$\mu_n = \mu$ and $\lambda_n = \lambda_0 + w^2 n\mu_0$, where w denotes the linking number of K and c.*

In the following, we denote $E(K)$ by V to emphasize that it is a solid torus. It should be noted that a meridian of K is a preferred longitude of V and a preferred longitude of K is a meridian of V. Then $E(K_n) = V(c; -\frac{1}{n})$.

Suppose that K_n is a graph knot, i.e., $E(K_n)$ is a graph manifold. If K_n is also a trivial knot, then from [20,18] we see that $|n| \leqslant 1$. So in the following we assume that K_n is nontrivial. Then each label appeared at vertices of satellite diagram of K_n is T, *Ca* or *Co*.

Assume first that the innermost vertex has a label T (i.e., K_n is a torus knot). Then if (K, D) is not an exceptional pair of type (ε_1, q_1), we have $|n| \leqslant 1$ [26, Theorem 3.8], see also [24].

Next suppose that the innermost vertex has a label *Co* (i.e., K_n is a composite knot). In this case, we can conclude that $|n| = 1$ from more general results in [6,14].

Thus in the following we assume that the innermost vertex has a label *Ca* (i.e., K_n is a cable knot). To make it precise, we assume that K_n is a (p,q)-cable of some graph knot k, where p and q are relatively prime and $q \geqslant 2$. Let t be a regular fiber of the cable space P which is a decomposing piece containing $\partial N(K_n)$. Then $t = pq\mu_n + \lambda_n$, which is written as $(pq + w^2 n)\mu_0 + \lambda_0$ by Claim 4.2.

Attach a solid torus W to V in such a way that the meridian of W is identified with a regular fiber t. Then we obtain a 3-manifold $V \cup W$ and denote the image of c in $V \cup W$ by c' to emphasize that it is in $V \cup W$. Since V is a solid torus, the manifold $V \cup W$ is homeomorphic to $S^2 \times S^1$ if $pq + w^2 n = 0$ (i.e., $t = \lambda_0$), S^3 if $|pq + w^2 n| = 1$ (i.e., $t = \pm\mu_0 + \lambda_0$), or a lens space $L(pq + w^2 n, 1)$ if $|pq + w^2 n| \geqslant 2$.

We denote the slope represented by a meridian of c by μ and the slope represented by $-1/n$ by γ. Since the meridian of c is also a meridian of c', we use the same symbol μ to denote the meridian of c'. For simplicity, we continue to use the same symbol γ to denote the corresponding slope for c'.

Lemma 4.3. *$(V \cup W)(c'; \gamma) = V(c;\gamma) \cup W$ is a reducible manifold without $S^2 \times S^1$ summand.*

Proof. Since $V(c;\gamma) = E(K_n)$, the manifold in question is obtained from $E(K_n)$ by attaching the solid W so that a meridian of W is identified with a regular fiber of the decomposing piece P. Hence the resulting manifold is $(K_n; pq) \cong (k; \frac{p}{q}) \sharp L$ for the companion knot k and some lens space $L \ncong S^3, S^2 \times S^1$, see [7]. Since $q \geqslant 2$, by [10], $(k; \frac{p}{q}) \ncong S^3$, hence $(V \cup W)(c'; \gamma) = V(c; \gamma) \cup W$ is reducible.

Since $H_1(V(c; \gamma) \cup W) \cong H_1((K_n; pq)) \cong \mathbb{Z}_{pq}$ is finite, $V(c; \gamma) \cup W$ does not contain a non-separating 2-sphere, in particular, it has no $S^2 \times S^1$-summand. $\square$

For two slopes γ_1 and γ_2 of a knot, the distance $\Delta(\gamma_1, \gamma_2)$ between them is defined to be their minimal geometric intersection number.

Lemma 4.4. *If* $pq + w^2 n = 0$, *then* $|n| = 1$.

Proof. Since $pq + w^2 n = 0$, $(V \cup W)(c'; \mu) = V(c; \mu) \cup W \cong V \cup W \cong S^2 \times S^1$. By Lemma 4.3, $(V \cup W)(c'; \gamma) = V(c; \gamma) \cup W$ is a reducible manifold without $S^2 \times S^1$ summand. If $V \cup W - \operatorname{int} N(c')$ is reducible, then the primeness of $S^2 \times S^1$ implies that c' is contained in a 3-ball in $V \cup W$. This means that $(V \cup W)(c'; \gamma)$ has $S^2 \times S^1$ as a summand, a contradiction. Hence $V \cup W - \operatorname{int} N(c')$ is irreducible. Apply [11] to conclude that $\Delta(\gamma, \mu) = 1$, i.e., the slope γ is integral and hence $|n| = 1$. $\square$

Lemma 4.5. *If* $|pq + w^2 n| = 1$, *then* $|n| = 1$.

Proof. Under this assumption, $V \cup W \cong S^3$. Since $(V \cup W)(c'; \gamma)$ is reducible (Lemma 4.3), by [9], $\Delta(\gamma, \mu) = 1$, i.e., the slope γ is integral and hence $|n| = 1$. $\square$

The rest of this section is devoted to prove:

Lemma 4.6. *Suppose that* (K, D) *is a hyperbolic pair and* $|pq + w^2 n| \geqslant 2$. *Then* $|n| = 1$.

Proof. For simplicity, set $X = V \cup W - \operatorname{int} N(c')$. Note that $V \cup W$ is a lens space $L(pq + w^2 n, 1)$. Let us now divide the proof into the following three cases:

(1) $X = L(pq + w^2 n, 1) - \operatorname{int} N(c')$ is irreducible and not an atoroidal Seifert fiber space.
(2) X is an atoroidal Seifert fiber space.
(3) X is reducible.

Recall that

- $(V \cup W)(c'; \mu) = V \cup W = L(pq + w^2 n, 1)$.
- $(V \cup W)(c'; \gamma) = V(c; \gamma) \cup W$ is a reducible manifold without $S^2 \times S^1$ summand (Lemma 4.3).

Case (1). Since μ is a cyclic surgery slope and γ is a reducing surgery slope for X, apply [2, Theorem 1.2] to conclude that $\Delta(\gamma, \mu) = 1$, i.e., $|n| = 1$, as desired.

Case (2). Since X is an atoroidal Seifert fiber space, the base orbifold is either the disk with at most two cone points or the Möbius band with no cone points. If the latter case occurs, then X is a twisted I-bundle over the Klein bottle, hence X admits also a Seifert fibration whose base orbifold is the disk with two cone points of indices $2, 2$. Thus the latter case reduces to the former case.

Now let us assume that the base orbifold of X is the disk with at most one cone point. Then X is a solid torus, and hence $L(pq + w^2n, 1)(c'; \gamma) = (V \cup W)(c'; \gamma)$ admits a genus one Heegaard splitting. This contradicts Lemma 4.3. It follows that the base orbifold of X is the disk with exactly two cone points. Let t be a slope represented by a regular fiber in $\partial N(c') \subset X$. Then $L(pq + w^2n, 1)(c'; \gamma) = (V \cup W)(c'; \gamma)$ is (i) a connected sum of two lens spaces if $\Delta(\gamma, t) = 0$, (ii) a lens space if $\Delta(\gamma, t) = 1$, or (iii) a Seifert fiber space over the 2-sphere with three exceptional fibers if $\Delta(\gamma, t) \geqslant 2$. A Seifert fiber space of type (iii) are neither lens space nor a reducible manifold [16, Example VI.13]. Thus $\Delta(\gamma, t) = 0$, i.e., $\gamma = t$. Since $\Delta(\mu, t) = 1$, we have $\Delta(\gamma, \mu) = 1$ as desired.

Case (3). Since a lens space $L(pq + w^2n, 1)$ is irreducible but $L(pq + w^2n, 1) - \operatorname{int} N(c')$ is reducible, c' is contained in a 3-ball $B \subset L(pq + w^2n, 1)$. Since $V - \operatorname{int} N(c)$ is irreducible, $\Sigma = \partial B$ is not contained in V. Hence we assume that Σ intersects the solid torus W with non-empty meridian disks of W. We further assume that $|\Sigma \cap W|$, the number of components of $\Sigma \cap W$, is minimal among 2-spheres bounding 3-balls which contain c. Since Σ separates $V \cup W$, $|\Sigma \cap W|$ is an even integer $\geqslant 2$. Set $S = \Sigma \cap (V - \operatorname{int} N(c))$, which is a planar surface.

Lemma 4.7. *If* $|\partial S| \geqslant 4$, *then* γ *is integral* (*i.e.*, $|n| = 1$).

Proof. Assume that $|\partial S| \geqslant 4$. Since Σ separates $L(pq + w^2n, 1) = V \cup W$, S also separates V. Cutting V along S, we obtain two 3-manifolds M_1 and M_2. Without loss of generality we may assume that $M_1 \supset c$. The minimality of $|\Sigma \cap W|$ assures that S is incompressible in both $M_1 - \operatorname{int} N(c)$ and M_2. There are two cases to consider: (1) S is incompressible in $M_1(c; \gamma)$, or (2) S is compressible in $M_1(c; \gamma)$.

(1) S is incompressible in $M_1(c; \gamma)$. Then S is incompressible in $V(c; \gamma) = M_1(c; \gamma) \cup_S M_2$. Since $|\partial S| \geqslant 4$, S is boundary-incompressible in $V(c; \gamma) \cong E(K_n)$. Recall also that a boundary component of S is lying on $\partial V = \partial E(K_n)$ and has slope $pq\mu_n + \lambda_n$. Then from Proposition 3.1 we see that S should be a cabling annulus, in particular $|\partial S| = 2$, a contradiction.

(2) S is compressible in $M_1(c; \gamma)$.

Claim 4.8. *S is compressible also in* $M_1 = M_1(c; \mu)$.

Proof. If S is incompressible in M_1, then S is also incompressible in $V = M_1 \cup_S M_2$. This implies that the solid torus V contains an incompressible planar surface S with $|\partial S| \geqslant 4$, a contradiction. □

Suppose that there is no incompressible annulus in $M_1 - \operatorname{int} N(c)$ with one boundary component in S and the other in $\partial N(c)$. Then Wu [31, Theorem 1] shows that $\Delta(\gamma, \mu) = 1$, i.e., γ is integral as claimed in Lemma 4.7.

Let us assume that there is such an annulus, say A, in $M_1 - \operatorname{int} N(c)$. Write $\partial A = C_1 \cup C_2$, where $C_1 \subset \partial N(c)$ and $C_2 \subset S(\subset \Sigma)$. Since C_2 bounds a disk in the 2-sphere Σ, C_1 bounds a disk in the 3-ball B. Thus c' is a trivial knot in B, and $\partial A \cap N(c)$ represents a longitudinal slope λ' of c'. Apply [3, Theorem 2.4.3(b)] to conclude that $\Delta(\gamma, \lambda') \leqslant 1$ or $M_1 - \operatorname{int} N(c) \cong S^1 \times S^1 \times I$. The latter implies that the incompressible surface S in $M_1 - \operatorname{int} N(c)$ is a disk or an annulus, contradicting the assumption $|\partial S| \geqslant 4$. It follows that $\Delta(\gamma, \lambda') \leqslant 1$. This, together with the triviality of $c' \subset B$, implies that either $B(c'; \gamma) = B(c'; 1/m) \cong B^3$ or $B(c'; \gamma) = B(c'; 0) \cong S^2 \times S^1$ with an open 3-ball removed. Hence $L(pq + w^2 n, 1)(c'; \gamma) = (L(pq + w^2 n, 1) - B) \cup B(c'; \gamma)$ is homeomorphic to $L(pq + w^2 n, 1)$ or $L(pq + w^2 n, 1) \sharp (S^2 \times S^1)$. This contradicts Lemma 4.3 and completes a proof of Lemma 4.7. □

To finish the proof of Lemma 4.6, assume for a contradiction that γ is not integral. Since $|\partial S|$ is even, Lemma 4.7 shows that $|\partial S| = 2$, i.e., S is an annulus. It follows that $S^3 - \operatorname{int} N(K \cup c) = V - \operatorname{int} N(c)$ contains an essential annulus. This contradicts the hyperbolicity of $S^3 - \operatorname{int} N(K \cup c) = V - \operatorname{int} N(c)$. □

Now the proof of Proposition 4.1 follows from Lemmas 4.4–4.6.

5. Proof of Theorem 1.1 for non-hyperbolic pairs

In this section we will prove Theorem 1.1 in the case where K and $c = \partial D$ forms a non-hyperbolic link.

Proposition 5.1. *Suppose that (K, D) is a non-hyperbolic pair and $K_{D,n}$ is a graph knot. Then $|n| \leqslant 1$ or (K, D) is an exceptional pair.*

Proof. If $S^3 - \operatorname{int} N(K \cup \partial D) = S^3 - \operatorname{int} N(K \cup c)$ is Seifert fibered, then (K, D) is an exceptional pair of type (ε_1, q_1).

Let us suppose that $S^3 - \operatorname{int} N(K \cup c)$ contains essential tori. Let $\mathcal{T}$ be a family of essential tori $T_1, \ldots, T_n$ which defines a torus decomposition of $S^3 - \operatorname{int} N(K \cup c)$ in the sense of Jaco and Shalen [15] and Johannson [17].

Lemma 5.2. *Each torus in $\mathcal{T}$ separates $\partial N(K)$ and $\partial N(c)$. Hence each decomposing piece has exactly two boundary components.*

Proof. Assume for a contradiction that there is a torus $T_i \in \mathcal{T}$ which does not separate $\partial N(K)$ and $\partial N(c)$. By the solid torus theorem [27], T_i bounds a solid torus V_i. Since T_i is incompressible in $S^3 - \operatorname{int} N(K \cup c)$, V_i is knotted in S^3 and contains both K and c in its interior. Furthermore, the triviality of K and c in S^3 implies that there are 3-balls B_K and B_c in V_i such that $K \subset B_K$ and $c \subset B_c$. Choose a meridian disk D of V_i so that $D \cap c = \emptyset$; an existence of the above 3-ball B_c assures an existence of such a meridian disk. Since $K \subset B_K$, the algebraic intersection number of K and D is zero. Moreover, since $D \cap c = \emptyset$, the algebraic intersection number of K_n and D, i.e., the winding number $\mathrm{wind}_{V_i}(K_n)$ of K_n in (the companion solid torus) V_i is still zero. This contradicts the following claim. □

Claim 5.3. *Let k be a graph knot and W a companion solid torus of k. Then the winding number of k in W is not zero.*

Proof. Let us consider the torus decomposition of $W - \operatorname{int} N(k)$. Choose the subfamily $\{S_1, \ldots, S_n\}$ consisting of tori each of which separates ∂W and $\partial N(k)$. Then we obtain solid tori W_i in W bounded by S_i so that $W \supset W_1 \supset \cdots \supset W_n \supset k$. Assume that $\mathrm{wind}_W(k) = 0$. Then since $\mathrm{wind}_W(k) = \mathrm{wind}_W(C_{W_1})\,\mathrm{wind}_{W_1}(C_{W_2}) \cdots \mathrm{wind}_{W_n}(k)$, where C_{W_i} denotes a core of W_i, at least one of $\mathrm{wind}_W(C_{W_1})$, $\mathrm{wind}_{W_1}(C_{W_2}), \ldots, \mathrm{wind}_{W_n}(k)$ is zero. Note that $W_j - \operatorname{int} W_{j+1}$ is a (p, q)-cable space or the union of a composing space P and some graph knot exteriors, where $\partial W_j, \partial W_{j+1} \subset \partial P$. In the former case, $\mathrm{wind}_{W_j}(C_{W_{j+1}}) = q \geqslant 2$, and in the latter case, $\mathrm{wind}_{W_j}(C_{W_{j+1}}) = 1$, a contradiction. □

Let T_1 be the (unique) innermost torus with respect to $\partial N(c)$, and let P be the decomposing piece bounded by T_1 and $\partial N(c)$.

Suppose first that P is hyperbolic. Cutting S^3 along T_1, we obtain two 3-manifolds $W(\supset K)$ and $W'(\supset c)$.

Claim 5.4. *W is an unknotted solid torus in S^3.*

Proof. By the solid torus theorem [27], W or W' is a solid torus. Assume that W (respectively W') is a solid torus. Since T_1 is incompressible in $S^3 - \operatorname{int} N(K \cup c)$, T_1 is incompressible also in $W - \operatorname{int} N(K)$ (respectively $W' - \operatorname{int} N(c)$). The nontriviality of K (respectively c) implies that W is unknotted (respectively W' is unknotted, and hence $W = S^3 - \operatorname{int} W'$ is also an unknotted solid torus). □

Let J be a core of W, then J is a trivial knot by Claim 5.4.

After $-\frac{1}{n}$-surgery on c, we obtain K_n and J_n as the images of K and J, respectively. Note that J_n is a companion knot of K_n and since K_n is a graph knot, J_n is also a graph knot. Since $S^3 - \operatorname{int} N(J \cup C) = S^3 - \operatorname{int}(W \cup N(C)) = P$ is hyperbolic, we can apply Proposition 4.1 to the pair J and c, and conclude that $|n| = 1$.

Now assume that P is Seifert fibered. Since ∂P consists of two components, P is a cable space, see Fig. 7 in which P is a $(1, 2)$-cable space.

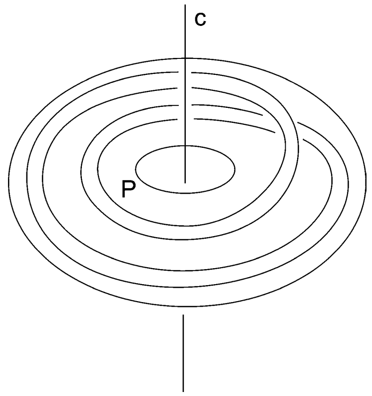

Fig. 7.

Then since K is unknotted in S^3, P is a $(\pm 1, q)$-cable space for some integer $q \geqslant 2$, and a regular fiber of P represents $q\mu_c \pm \lambda_c$ in terms of a preferred meridian-longitude pair (μ_c, λ_c) of c.

Recall that $\mathcal{T}$ is a family of essential tori defining the torus decomposition of $S^3 - \operatorname{int} N(K \cup c)$.

Claim 5.5. *If $|n| > 1$, then the family $\mathcal{T}$ defines also a torus decomposition of*

$$E(K_n) = \big(S^3 - \operatorname{int} N(K \cup c)\big) \cup_{-\frac{1}{n}} N(c).$$

Proof. Let us consider $P \cup_{-\frac{1}{n}} N(c)$. Since $q \geqslant 2$ and $|n| > 1$, $\Delta(\pm q, -\frac{1}{n}) = |\pm nq + 1| \geqslant 3$. Thus the Seifert fibration of P can be extended to that of $P \cup_{-\frac{1}{n}} N(c)$ over the disk with two exceptional fibers of indices q, $|qn + \varepsilon|$ $(\varepsilon = \pm 1)$. Hence it is boundary-irreducible and admits a unique Seifert fibration up to isotopy. It turns out that $\mathcal{T}$ defines also the torus decomposition of $E(K_n) = (S^3 - \operatorname{int} N(K \cup c)) \cup_{-\frac{1}{n}} N(c)$. □

Let $P_1 = P, P_2, \ldots, P_m$ be decomposing pieces of $S^3 - \operatorname{int} N(K \cup c)$. By Claim 5.2 each P_i has exactly two boundary components. From Claim 5.5, we see that $P_1 \cup_{-\frac{1}{n}} N(c), P_2, \ldots, P_m$ are decomposing pieces of $E(K_n) = (S^3 - \operatorname{int} N(K \cup c)) \cup_{-\frac{1}{n}} N(c)$.

Since K_n is a graph knot, $P_2, \ldots, P_m$ are Seifert fiber spaces. Since each P_i has exactly two boundary components, P_i is a cable space. The triviality of K in S^3 implies that P_i is a (ε_i, q_i)-cable space, where $\varepsilon_i = \pm 1$ and $q_i \geqslant 2$. It follows that (K, D) is an exceptional pair as desired. □

Theorem 1.1 follows from Propositions 4.1 and 5.1.

We close this paper by noting a relationship between Proposition 4.1 and surgeries on knots in a solid torus. In [12] Gordon and Luecke proved that a toroidal surgery on a hyperbolic knot in a solid torus is integral or half-integral. If a surgery on a hyperbolic knot in a solid torus yields a Seifert fiber space, then the surgery is integral [24]. Is a surgery on a hyperbolic knot in a solid torus producing a graph manifold also integral? If this is true, then Proposition 4.1 follows in this direction. However, there are infinitely many non-integral (half-integral) surgeries on hyperbolic knots in a solid torus producing graph manifolds, see [23].

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