

3

Matrix Algebra and Applications

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CASE STUDY The Japanese Economy

A senator walks into your cubicle in the Congressional Budget Office. “Look here,” she says, “I don’t see why the Japanese trade representative is getting so upset with my proposal to cut down on our use of Japanese finance and insurance. He claims that it’ll hurt Japan’s mining operations. But just look at Japan’s input-output table. The finance sector doesn’t use any input from the mining sector. How can our cutting down demand for finance and insurance hurt mining?” How should you respond?



Jose Fuste-Ranga/Zefar/Corbis

Online you will find:

- Section by section tutorials
- A detailed chapter summary
- A true/false quiz
- Additional review exercises
- A matrix algebra tool, game theory utility, and other resources

Introduction

We used matrices in Chapter 2 simply to organize our work. It is time we examined them as interesting objects in their own right. There is much that we can do with matrices besides row operations: We can add, subtract, multiply, and even, in a sense, “divide” matrices. We use these operations to study game theory and input-output models in this chapter, and Markov chains in a later chapter.

Many calculators, electronic spreadsheets, and other computer programs can do these matrix operations, which is a big help in doing calculations. However, we need to know how these operations are defined to see why they are useful and to understand which to use in any particular application.

3.1 Matrix Addition and Scalar Multiplication

Let’s start by formally defining what a matrix is and introducing some basic terms.

Matrix, Dimension, and Entries

An $m \times n$ matrix A is a rectangular array of real numbers with m rows and n columns. We refer to m and n as the **dimensions** of the matrix. The numbers that appear in the matrix are called its **entries**. We customarily use capital letters A, B, C, \dots for the names of matrices.

quick Examples

1. $A = \begin{bmatrix} 2 & 0 & 1 \\ 33 & -22 & 0 \end{bmatrix}$ is a 2×3 matrix because it has 2 rows and 3 columns.

2. $B = \begin{bmatrix} 2 & 3 \\ 10 & 44 \\ -1 & 3 \\ 8 & 3 \end{bmatrix}$ is a 4×2 matrix because it has 4 rows and 2 columns.

The entries of A are 2, 0, 1, 33, -22 , and 0. The entries of B are the numbers 2, 3, 10, 44, -1 , 3, 8, and 3.

Hint: Remember that the number of rows is given first and the number of columns second. An easy way to remember this is to think of the acronym “RC” for “Row then Column.”

Referring to the Entries of a Matrix

There is a systematic way of referring to particular entries in a matrix. If i and j are numbers, then the entry in the i th row and j th column of the matrix A is called the **ij th entry** of A . We usually write this entry as a_{ij} or A_{ij} . (If the matrix was called B , we would write its ij th entry as b_{ij} or B_{ij} .) Notice that this follows the “RC” convention: The row number is specified first and the column number second.

quick Example

With $A = \begin{bmatrix} 2 & 0 & 1 \\ 33 & -22 & 0 \end{bmatrix}$,

$$a_{13} = 1$$

First row, third column

$$a_{21} = 33$$

Second row, first column

**using Technology**

See the Technology Guides at the end of the chapter to see how matrices are entered and used in a TI-83/84 or Excel. For the authors' web-based utility, follow:

Chapter 3
→ Tools
→ Matrix Algebra Tool

There you will find a computational tool that allows you to do matrix algebra. Use the following format to enter the matrix A on the previous page (spaces are optional):

$$A = [2, 0, 1 \\ 33, -22, 0]$$

To display the matrix A , type A in the formula box and press "Compute."

According to the labeling convention, the entries of the matrix A above are

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

In general, the $m \times n$ matrix A has its entries labeled as follows:

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$$

We say that two matrices A and B are **equal** if they have the same dimensions and the corresponding entries are equal. Note that a 3×4 matrix can never equal a 3×5 matrix because they do not have the same dimensions.

Example 1 Matrix Equality

Let $A = \begin{bmatrix} 7 & 9 & x \\ 0 & -1 & y+1 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & 9 & 0 \\ 0 & -1 & 11 \end{bmatrix}$. Find the values of x and y such that $A = B$.

Solution For the two matrices to be equal, we must have corresponding entries equal, so

$$\begin{array}{lcl} x & = & 0 \\ y + 1 = 11 & \text{or} & y = 10 \end{array} \qquad \begin{array}{l} a_{13} = b_{13} \\ a_{23} = b_{23} \end{array}$$

† *Before we go on...* Note in Example 1 that the matrix equation

$$\begin{bmatrix} 7 & 9 & x \\ 0 & -1 & y+1 \end{bmatrix} = \begin{bmatrix} 7 & 9 & 0 \\ 0 & -1 & 11 \end{bmatrix}$$

is really six equations in one: $7 = 7$, $9 = 9$, $x = 0$, $0 = 0$, $-1 = -1$, and $y + 1 = 11$. We used only the two that were interesting. ■

Row Matrix, Column Matrix, and Square Matrix

A matrix with a single row is called a **row matrix**, or **row vector**. A matrix with a single column is called a **column matrix** or **column vector**. A matrix with the same number of rows as columns is called a **square matrix**.

quick Examples

The 1×5 matrix $C = [3 \quad -4 \quad 0 \quad 1 \quad -11]$ is a row matrix.

The 4×1 matrix $D = \begin{bmatrix} 2 \\ 10 \\ -1 \\ 8 \end{bmatrix}$ is a column matrix.

The 3×3 matrix $E = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 4 \\ -4 & 32 & 1 \end{bmatrix}$ is a square matrix.

Matrix Addition and Subtraction

The first matrix operations we discuss are matrix addition and subtraction. The rules for these operations are simple.

Matrix Addition and Subtraction

Two matrices can be added (or subtracted) if and only if they have the same dimensions. To add (or subtract) two matrices of the same dimensions, we add (or subtract) the corresponding entries. More formally, if A and B are $m \times n$ matrices, then $A + B$ and $A - B$ are the $m \times n$ matrices whose entries are given by:

$$(A + B)_{ij} = A_{ij} + B_{ij} \quad ij \text{th entry of the sum} = \text{sum of the } ij \text{th entries}$$

$$(A - B)_{ij} = A_{ij} - B_{ij} \quad ij \text{th entry of the difference} = \text{difference of the } ij \text{th entries}$$

Visualizing Matrix Addition

$$\begin{bmatrix} 2 & -3 \\ 1 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 3 & -2 \\ -1 & 1 \end{bmatrix}$$

quick Examples

$$1. \begin{bmatrix} 2 & -3 \\ 1 & 0 \\ -1 & 3 \end{bmatrix} + \begin{bmatrix} 9 & -5 \\ 0 & 13 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} 11 & -8 \\ 1 & 13 \\ -2 & 6 \end{bmatrix} \quad \text{Corresponding entries added}$$

$$2. \begin{bmatrix} 2 & -3 \\ 1 & 0 \\ -1 & 3 \end{bmatrix} - \begin{bmatrix} 9 & -5 \\ 0 & 13 \\ -1 & 3 \end{bmatrix} = \begin{bmatrix} -7 & 2 \\ 1 & -13 \\ 0 & 0 \end{bmatrix} \quad \text{Corresponding entries subtracted}$$

Example 2 Sales

The A-Plus auto parts store chain has two outlets, one in Vancouver and one in Quebec. Among other things, it sells wiper blades, windshield cleaning fluid, and floor mats. The monthly sales of these items at the two stores for two months are given in the following tables:

January Sales

	Vancouver	Quebec
Wiper Blades	20	15
Cleaning Fluid (bottles)	10	12
Floor Mats	8	4

February Sales

	Vancouver	Quebec
Wiper Blades	23	12
Cleaning Fluid (bottles)	8	12
Floor Mats	4	5

**using Technology**

See the Technology Guides at the end of the chapter to see how to add and subtract matrices using a TI-83/84 or Excel. Alternatively, use the Matrix Algebra Tool at

Chapter 3

→ Tools

→ Matrix Algebra Tool

There, first enter the two matrices you wish to add or subtract (subtract, in this case) as shown:

$$J = [20, 15, 10, 12, 8, 4]$$

$$F = [23, 12, 8, 12, 4, 5]$$

To compute their difference, type $F - J$ in the formula box and press “Compute.” (You can enter multiple formulas separated by commas in the formula box. For instance, $F + J$, $F - J$ will compute both the sum and difference.)

Use matrix arithmetic to calculate the change in sales of each product in each store from January to February.

Solution The tables suggest two matrices:

$$J = \begin{bmatrix} 20 & 15 \\ 10 & 12 \\ 8 & 4 \end{bmatrix} \quad \text{and} \quad F = \begin{bmatrix} 23 & 12 \\ 8 & 12 \\ 4 & 5 \end{bmatrix}$$

To compute the change in sales of each product for both stores, we want to subtract corresponding entries in these two matrices. In other words, we want to compute the difference of the two matrices:

$$F - J = \begin{bmatrix} 23 & 12 \\ 8 & 12 \\ 4 & 5 \end{bmatrix} - \begin{bmatrix} 20 & 15 \\ 10 & 12 \\ 8 & 4 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ -2 & 0 \\ -4 & 1 \end{bmatrix}$$

Thus, the change in sales of each product is the following:

	Vancouver	Quebec
Wiper Blades	3	-3
Cleaning Fluid (bottles)	-2	0
Floor Mats	-4	1

Scalar Multiplication

A matrix A can be added to itself because the expression $A + A$ is the sum of two matrices that have the same dimensions. When we compute $A + A$, we end up doubling every entry in A . So we can think of the expression $2A$ as telling us to *multiply every element in A by 2*.

In general, to multiply a matrix by a number, multiply every entry in the matrix by that number. For example,

$$6 \begin{bmatrix} \frac{5}{2} & -3 \\ 1 & 0 \\ -1 & \frac{5}{6} \end{bmatrix} = \begin{bmatrix} 15 & -18 \\ 6 & 0 \\ -6 & 5 \end{bmatrix}$$

It is traditional when talking about matrices to call individual numbers **scalars**. For this reason, we call the operation of multiplying a matrix by a number **scalar multiplication**.

Example 3 Sales

The revenue generated by sales in the Vancouver and Quebec branches of the A-Plus auto parts store (see Example 2) was as follows:

January Sales in Canadian Dollars

	Vancouver	Quebec
Wiper Blades	140.00	105.00
Cleaning Fluid	30.00	36.00
Floor Mats	96.00	48.00

**using Technology**

See the Technology Guides at the end of the chapter to see how to compute scalar multiples using a TI-83/84 or Excel. Alternatively, go to the online Matrix Algebra Tool at

Chapter 3

→ Tools

→ Matrix Algebra Tool

There, enter the January sales in U.S. Dollars:

$$A = [140, 105 \\ 30, 36 \\ 96, 48]$$

Then type $0.65 * A$ in the formula box and press “Compute.”

If the Canadian dollar was worth \$0.65 U.S. at the time, compute the revenue in U.S. dollars.

Solution We need to multiply each revenue figure by 0.65. Let A be the matrix of revenue figures in Canadian dollars:

$$A = \begin{bmatrix} 140.00 & 105.00 \\ 30.00 & 36.00 \\ 96.00 & 48.00 \end{bmatrix}$$

The revenue figures in U.S. dollars are then given by the scalar multiple

$$0.65A = 0.65 \begin{bmatrix} 140.00 & 105.00 \\ 30.00 & 36.00 \\ 96.00 & 48.00 \end{bmatrix} = \begin{bmatrix} 91.00 & 68.25 \\ 19.50 & 23.40 \\ 62.40 & 31.20 \end{bmatrix}$$

In other words, in U.S. dollars, \$91 worth of wiper blades was sold in Vancouver, \$68.25 worth of wiper blades was sold in Quebec, and so on.

Formally, scalar multiplication is defined as follows:

Scalar Multiplication

If A is an $m \times n$ matrix and c is a real number, then cA is the $m \times n$ matrix obtained by multiplying all the entries of A by c . (We usually use lowercase letters c, d, e, \dots to denote scalars.) Thus, the ij th entry of cA is given by

$$(cA)_{ij} = c(A_{ij})$$

In words, this rule is: To get the ij th entry of cA , multiply the ij th entry of A by c .

Example 4 Combining Operations

Let $A = \begin{bmatrix} 2 & -1 & 0 \\ 3 & 5 & -3 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 3 & -1 \\ 5 & -6 & 0 \end{bmatrix}$, and $C = \begin{bmatrix} x & y & w \\ z & t+1 & 3 \end{bmatrix}$

Evaluate the following: $4A$, xB , and $A + 3C$.

Solution First, we find $4A$ by multiplying each entry of A by 4:

$$4A = 4 \begin{bmatrix} 2 & -1 & 0 \\ 3 & 5 & -3 \end{bmatrix} = \begin{bmatrix} 8 & -4 & 0 \\ 12 & 20 & -12 \end{bmatrix}$$

Similarly, we find xB by multiplying each entry of B by x :

$$xB = x \begin{bmatrix} 1 & 3 & -1 \\ 5 & -6 & 0 \end{bmatrix} = \begin{bmatrix} x & 3x & -x \\ 5x & -6x & 0 \end{bmatrix}$$

We get $A + 3C$ in two steps as follows:

$$A + 3C = \begin{bmatrix} 2 & -1 & 0 \\ 3 & 5 & -3 \end{bmatrix} + 3 \begin{bmatrix} x & y & w \\ z & t+1 & 3 \end{bmatrix}$$

$$\begin{aligned}
 &= \begin{bmatrix} 2 & -1 & 0 \\ 3 & 5 & -3 \end{bmatrix} + \begin{bmatrix} 3x & 3y & 3w \\ 3z & 3t+3 & 9 \end{bmatrix} \\
 &= \begin{bmatrix} 2+3x & -1+3y & 3w \\ 3+3z & 3t+8 & 6 \end{bmatrix}
 \end{aligned}$$

Addition and scalar multiplication of matrices have nice properties, reminiscent of the properties of addition and multiplication of real numbers. Before we state them, we need to introduce some more notation.

If A is any matrix, then $-A$ is the matrix $(-1)A$. In other words, $-A$ is A multiplied by the scalar -1 . This amounts to changing the signs of all the entries in A . For example,

$$-\begin{bmatrix} 4 & -2 & 0 \\ 6 & 10 & -6 \end{bmatrix} = \begin{bmatrix} -4 & 2 & 0 \\ -6 & -10 & 6 \end{bmatrix}$$

For any two matrices A and B , $A - B$ is the same as $A + (-B)$. (Why?)

Also, a **zero matrix** is a matrix all of whose entries are zero. Thus, for example, the 2×3 zero matrix is

$$O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Now we state the most important properties of the operations that we have been talking about:

Properties of Matrix Addition and Scalar Multiplication

If A , B , and C are any $m \times n$ matrices and if O is the zero $m \times n$ matrix, then the following hold:

$A + (B + C) = (A + B) + C$	<i>Associative law</i>
$A + B = B + A$	<i>Commutative law</i>
$A + O = O + A = A$	<i>Additive identity law</i>
$A + (-A) = O = (-A) + A$	<i>Additive inverse law</i>
$c(A + B) = cA + cB$	<i>Distributive law</i>
$(c + d)A = cA + dA$	<i>Distributive law</i>
$1A = A$	<i>Scalar unit</i>
$0A = O$	<i>Scalar zero</i>

These properties would be obvious if we were talking about addition and multiplication of *numbers*, but here we are talking about addition and multiplication of *matrices*. We are using “+” to mean something new: matrix addition. There is no reason why matrix addition has to obey *all* the properties of addition of numbers. It happens that it does obey many of them, which is why it is convenient to call it *addition* in the first place. This means that we can manipulate equations involving matrices in much the same way that we manipulate equations involving numbers. One word of caution: We haven’t yet discussed how to multiply matrices, and it probably isn’t what you think. It will turn out that multiplication of matrices does *not* obey all the same properties as multiplication of numbers.

Transposition

We mention one more operation on matrices:

Transposition

If A is an $m \times n$ matrix, then its **transpose** is the $n \times m$ matrix obtained by writing its rows as columns, so that the i th row of the original matrix becomes the i th column of the transpose. We denote the transpose of the matrix A by A^T .

Visualizing Transposition

$$\begin{bmatrix} 2 & -3 \\ 1 & 0 \\ 5 & 1 \end{bmatrix} \quad \begin{bmatrix} 2 & 1 & 5 \\ -3 & 0 & 1 \end{bmatrix}$$

quick Examples

1. Let $B = \begin{bmatrix} 2 & 3 \\ 10 & 44 \\ -1 & 3 \\ 8 & 3 \end{bmatrix}$. Then $B^T = \begin{bmatrix} 2 & 10 & -1 & 8 \\ 3 & 44 & 3 & 3 \end{bmatrix}$.

4×2 matrix 2×4 matrix

2. $[-1 \ 1 \ 2]^T = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$

1×3 matrix 3×1 matrix



using Technology

See the Technology Guides at the end of the chapter to see how to transpose a matrix using a TI-83/84 or Excel. Alternatively, go to the online Matrix Algebra Tool at

Chapter 3

→ Tools

→ Matrix Algebra Tool

There, first enter the matrix you wish to transpose:

$$A = [2, 0, 1 \\ 33, -22, 0]$$

Then type A^T in the formula box and press “Compute.”

Properties of Transposition

If A and B are $m \times n$ matrices, then the following hold:

$$(A + B)^T = A^T + B^T$$

$$(cA)^T = c(A^T)$$

$$(A^T)^T = A$$

To see why the laws of transposition are true, let us consider the first one: $(A + B)^T = A^T + B^T$. The left-hand side is the transpose of $A + B$, and so is obtained by first adding A and B , and then writing the rows as columns. This is the same as first writing the rows of A and B individually as columns before adding, which gives the right-hand side. Similar arguments can be used to establish the other laws of transposition.

3.1 EXERCISES

● denotes basic skills exercises

tech Ex indicates exercises that should be solved using technology

In each of Exercises 1–10, find the dimensions of the given matrix and identify the given entry.

$$1. \bullet A = \begin{bmatrix} 1 & 5 & 0 & \frac{1}{4} \end{bmatrix}; a_{13} \quad 2. \bullet B = \begin{bmatrix} 44 & 55 \end{bmatrix}; b_{12}$$

$$3. \bullet C = \begin{bmatrix} \frac{5}{2} \\ 1 \\ -2 \\ 8 \end{bmatrix}; c_{11} \quad 4. \bullet D = \begin{bmatrix} 15 & -18 \\ 6 & 0 \\ -6 & 5 \\ 48 & 18 \end{bmatrix}; d_{31}$$

$$5. \bullet E = \begin{bmatrix} e_{11} & e_{12} & e_{13} & \cdots & e_{1q} \\ e_{21} & e_{22} & e_{23} & \cdots & e_{2q} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ e_{p1} & e_{p2} & e_{p3} & \cdots & e_{pq} \end{bmatrix}; E_{22}$$

$$6. \bullet A = \begin{bmatrix} 2 & -1 & 0 \\ 3 & 5 & -3 \end{bmatrix}; A_{21} \quad 7. \bullet B = \begin{bmatrix} 1 & 3 \\ 5 & -6 \end{bmatrix}; b_{12}$$

$$8. \bullet C = \begin{bmatrix} x & y & w & e \\ z & t+1 & 3 & 0 \end{bmatrix}; C_{23}$$

$$9. \bullet D = [d_1 \ d_2 \ \cdots \ d_n]; D_{1r} \text{ (any } r\text{)}$$

$$10. \bullet E = [d \ d \ d \ d]; E_{1r} \text{ (any } r\text{)}$$

11. ● Solve for x , y , z , and w . *hint* [see Example 1]

$$\begin{bmatrix} x+y & x+z \\ y+z & w \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 5 & 4 \end{bmatrix}$$

12. ● Solve for x , y , z , and w .

$$\begin{bmatrix} x-y & x-z \\ y-w & w \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 6 \end{bmatrix}$$

$$\text{In Exercises 13–20, let } A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \\ -1 & 2 \end{bmatrix}, B = \begin{bmatrix} 0.25 & -1 \\ 0 & 0.5 \\ -1 & 3 \end{bmatrix},$$

$$\text{and } C = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ -1 & -1 \end{bmatrix}. \text{ Evaluate:}$$

hint [see Quick Examples on pp. 176 and 180, and Example 4]

$$13. \bullet A + B \quad 14. \bullet A - C \quad 15. \bullet A + B - C$$

$$16. \bullet 12B \quad 17. \bullet 2A - C \quad 18. \bullet 2A + 0.5C$$

$$19. \bullet 2A^T \quad 20. \bullet A^T + 3C^T$$

$$\text{In Exercises 21–28, let } A = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 2 & -1 \end{bmatrix},$$

$$B = \begin{bmatrix} 3 & 0 & -1 \\ 5 & -1 & 1 \end{bmatrix}, \text{ and } C = \begin{bmatrix} x & 1 & w \\ z & r & 4 \end{bmatrix}. \text{ Evaluate:}$$

$$21. \bullet A + B \quad 22. \bullet B - C \quad 23. \bullet A - B + C$$

$$24. \bullet \frac{1}{2}B$$

$$25. \bullet 2A - B$$

$$26. \bullet 2A - 4C$$

$$27. \bullet 3B^T$$

$$28. \bullet 2A^T - C^T$$

tech Ex Use technology in Exercises 29–36.

$$\text{Let } A = \begin{bmatrix} 1.5 & -2.35 & 5.6 \\ 44.2 & 0 & 12.2 \end{bmatrix}, B = \begin{bmatrix} 1.4 & 7.8 \\ 5.4 & 0 \\ 5.6 & 6.6 \end{bmatrix},$$

$$C = \begin{bmatrix} 10 & 20 & 30 \\ -10 & -20 & -30 \end{bmatrix} \text{ Evaluate:}$$

$$29. \bullet A - C$$

$$30. \bullet C - A$$

$$31. \bullet 1.1B$$

$$32. \bullet -0.2B$$

$$33. \bullet A^T + 4.2B$$

$$34. \bullet (A + 2.3C)^T$$

$$35. \bullet (2.1A - 2.3C)^T \quad 36. \bullet (A - C)^T - B$$

Applications

37. ● **Cost of Real Estate** The following table shows the cost of one square foot of residential real estate, in dollars per square foot, at the start of 2001, and the changes in 2002 and 2003.¹

	New York	London	Hong Kong
2001	600	620	400
Change in 2002	120	60	-50
Change in 2003	40	120	-50

Use matrix algebra to find the cost of residential real estate in each city in **a.** 2002 and **b.** 2003. *hint* [see Example 2]

38. ● **Army Personnel** The following table shows the number of personnel, in thousands, in three branches of the U.S. Army in 2001, and the changes in 2002 and 2003.²

	Active Duty	Reserve	National Guard
2001	75	35	60
Change in 2002	5	-15	2
Change in 2003	-12	5	-17

Use matrix algebra to find to find the number of personnel in each branch in **a.** 2002 and **b.** 2003.

39. ● **Inventory** The Left Coast Bookstore chain has two stores, one in San Francisco and one in Los Angeles. It stocks three kinds of book: hardcover, softcover, and plastic (for infants).

¹ Figures are rounded. SOURCE: CLSA, Honk Kong Monetary Authority/*New York Times*, August 15, 2003, p. C1.

² Figures are rounded. SOURCE: Army Recruiting Command, Army National Guard Bureau/*New York Times*, September 22, 2003, p. A14.

● basic skills

tech Ex technology exercise

At the beginning of January, the central computer showed the following books in stock:

	Hard	Soft	Plastic
San Francisco	1000	2000	5000
Los Angeles	1000	5000	2000

Suppose its sales in January were as follows: 700 hardcover books, 1300 softcover books, and 2000 plastic books sold in San Francisco, and 400 hardcover, 300 softcover, and 500 plastic books sold in Los Angeles. Write these sales figures in the form of a matrix, and then show how matrix algebra can be used to compute the inventory remaining in each store at the end of January.

40. ● **Inventory** The Left Coast Bookstore chain discussed in Exercise 39 actually maintained the same sales figures for the first 6 months of the year. Each month, the chain restocked the stores from its warehouse by shipping 600 hardcover, 1500 softcover, and 1500 plastic books to San Francisco and 500 hardcover, 500 softcover, and 500 plastic books to Los Angeles.
- Use matrix operations to determine the total sales over the 6 months, broken down by store and type of book.
 - Use matrix operations to determine the inventory in each store at the end of June.
41. ● **Profit** Annual revenues and production costs at Luddington's Wellington Boots & Co. are shown in the following spreadsheet.

	2004	2006	2008
Revenue			
Full Boots	\$10,000	\$9,000	\$11,000
Half Boots	\$8,000	\$7,200	\$8,800
Sandals	\$4,000	\$5,000	\$6,000
Production Costs			
Full Boots	\$2,000	\$1,800	\$2,200
Half Boots	\$2,400	\$1,440	\$1,760
Sandals	\$1,200	\$1,500	\$2,000

Use matrix algebra to compute the profits from each sector each year.

42. ● **Revenue** The following spreadsheet gives annual production costs and profits at Gauss Jordan Sneakers, Inc.

	2004	2006	2008
Production Costs			
Gauss Grip	\$1,800	\$2,200	\$2,400
Air Gauss	\$1,400	\$1,700	\$1,200
Gauss Gel	\$1,500	\$2,000	\$1,300
Profit			
Gauss Grip	\$10,000	\$14,000	\$16,000
Air Gauss	\$8,000	\$12,000	\$14,000
Gauss Gel	\$9,000	\$14,000	\$12,000

Use matrix algebra to compute the revenues from each sector each year.

43. ● **Population Distribution** In 1980 the U.S. population, broken down by regions, was 49.1 million in the Northeast, 58.9 million in the Midwest, 75.4 million in the South, and 43.2 million in the West.³ In 1990 the population was 50.8 million in the Northeast, 59.7 million in the Midwest, 85.4 million in the South, and 52.8 million in the West. Set up the population figures for each year as a row vector, and then show how to use matrix operations to find the net increase or decrease of population in each region from 1980 to 1990.
44. ● **Population Distribution** In 1990 the U.S. population, broken down by regions, was 50.8 million in the Northeast, 59.7 million in the Midwest, 85.4 million in the South, and 52.8 million in the West.⁴ Between 1990 and 2000, the population in the Northeast grew by 2.8 million, the population in the Midwest grew by 4.7 million, the population in the South grew by 14.8 million, and the population in the West grew by 10.4 million. Set up the population figures for 1990 and the growth figures for the decade as row vectors. Assuming that the population will grow by the same numbers from 2000 to 2010 as they did from 1990 to 2000, show how to use matrix operations to find the population in each region in 2010.

Bankruptcy Filings Exercises 45–48 are based on the following chart, which shows the numbers of personal bankruptcy filings in three New York/New Jersey regions during various months of 2001–2002.⁵

	Jan 01	Apr	Jul	Oct	Jan 02
Manhattan	150	250	150	100	150
Brooklyn	300	400	300	200	250
Newark	250	400	250	200	200

45. ● Use matrix algebra to determine the total number of bankruptcy filings in each of the given months.
46. ● Use matrix algebra to determine the total number of bankruptcy filings in each of the given regions.
47. Use matrix algebra to determine in which month the difference between the number of bankruptcy filings in Brooklyn and in Newark was greatest.
48. Use matrix algebra to determine in which region the difference between the bankruptcy filings in April 2001 and January 2001 was greatest.
49. **Inventory** Microbucks Computer Company makes two computers, the Pomegranate II and the Pomegranate Classic, at

³ SOURCE: U.S. Census Bureau. Statistical Abstract of the United States: 2001. <http://www.census.gov>.

⁴ Ibid.

⁵ Data are approximate four-week moving averages. SOURCE: Lundquist Consulting/*New York Times*, February 10, 2002, p. L1.

two different factories. The Pom II requires 2 processor chips, 16 memory chips, and 20 vacuum tubes, while the Pom Classic requires 1 processor chip, 4 memory chips, and 40 vacuum tubes. Microbucks has in stock at the beginning of the year 500 processor chips, 5000 memory chips, and 10,000 vacuum tubes at the Pom II factory, and 200 processor chips, 2000 memory chips, and 20,000 vacuum tubes at the Pom Classic factory. It manufactures 50 Pom II's and 50 Pom Classics each month.

- Find the company's inventory of parts after two months, using matrix operations.
- When (if ever) will the company run out of one of the parts?

50. Inventory Microbucks Computer Company, besides having the stock mentioned in Exercise 49, gets shipments of parts every month in the amounts of 100 processor chips, 1000 memory chips, and 3000 vacuum tubes at the Pom II factory, and 50 processor chips, 1000 memory chips, and 2000 vacuum tubes at the Pom Classic factory.

- What will the company's inventory of parts be after six months?
- When (if ever) will the company run out of one of the parts?

51. Tourism The following table gives the number of people (in thousands) who visited Australia and South Africa in 1998:⁶

		To	Australia	South Africa
From	North America		440	190
	Europe		950	950
	Asia		1790	200

You predict that in 2008, 20,000 fewer people from North America will visit Australia and 40,000 more will visit South Africa, 50,000 more people from Europe will visit each of Australia and South Africa, and 100,000 more people from Asia will visit South Africa, but there will be no change in the number visiting Australia.

- Use matrix algebra to predict the number of visitors from the three regions to Australia and South Africa in 2008.
- Take A to be the 3×2 matrix whose entries are the 1998 tourism figures and take B to be the 3×2 matrix whose entries are the 2008 tourism figures. Give a formula (in terms of A and B) that predicts the average of the numbers of visitors from the three regions to Australia and South Africa in 1998 and 2008. Compute its value.

⁶ Figures are rounded to the nearest 10,000. SOURCES: South African Dept. of Environmental Affairs and Tourism; Australia Tourist Commission/ *The New York Times*, January 15, 2000, p. C1.

52. Tourism Referring to the 1998 tourism figures given in the preceding exercise, assume that the following (fictitious) figures represent the corresponding numbers from 1988.

		To	Australia	South Africa
From	North America		500	100
	Europe		900	800
	Asia		1400	50

Take A to be the 3×2 matrix whose entries are the 1998 tourism figures and take B to be the 3×2 matrix whose entries are the 1988 tourism figures.

- Compute the matrix $A - B$. What does this matrix represent?
- Assuming that the changes in tourism over 1988–1998 are repeated in 1998–2008, give a formula (in terms of A and B) that predicts the number of visitors from the three regions to Australia and South Africa in 2008.

Communication and Reasoning Exercises

- What does it mean when we say that $(A + B)_{ij} = A_{ij} + B_{ij}$?
- What does it mean when we say that $(cA)_{ij} = c(A_{ij})$?
- What would a 5×5 matrix A look like if $A_{ii} = 0$ for every i ?
- What would a matrix A look like if $A_{ij} = 0$ whenever $i \neq j$?
- Give a formula for the ij th entry of the transpose of a matrix A .
- A matrix is **symmetric** if it is equal to its transpose. Give an example of **a.** a nonzero 2×2 symmetric matrix and **b.** a nonzero 3×3 symmetric matrix.
- A matrix is **skew-symmetric** or **antisymmetric** if it is equal to the negative of its transpose. Give an example of **a.** a nonzero 2×2 skew-symmetric matrix and **b.** a nonzero 3×3 skew-symmetric matrix.
- Referring to Exercises 58 and 59, what can be said about a matrix that is both symmetric and skew-symmetric?
- Why is matrix addition associative?
- Describe a scenario (possibly based on one of the preceding examples or exercises) in which you might wish to compute $A - 2B$ for certain matrices A and B .
- Describe a scenario (possibly based on one of the preceding examples or exercises) in which you might wish to compute $A + B - C$ for certain matrices A , B and C .

3.2 Matrix Multiplication

Suppose we buy 2 CDs at \$3 each and 4 Zip disks at \$5 each. We calculate our total cost by computing the products' price \times quantity and adding:

$$\text{Cost} = 3 \times 2 + 5 \times 4 = \$26$$

Let us instead put the prices in a row vector

$$P = [3 \quad 5] \quad \text{The price matrix}$$

and the quantities purchased in a column vector,

$$Q = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \quad \text{The quantity matrix}$$

Q: Why a row and a column?

A: It's rather a long story, but mathematicians found that it works best this way . . .

Because P represents the prices of the items we are purchasing and Q represents the quantities, it would be useful if the product PQ represented the total cost, a *single number* (which we can think of as a 1×1 matrix). For this to work, PQ should be calculated the same way we calculated the total cost:



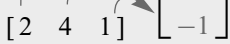
$$PQ = [3 \quad 5] \begin{bmatrix} 2 \\ 4 \end{bmatrix} = [3 \times 2 + 5 \times 4] = [26]$$

Notice that we obtain the answer by multiplying each entry in P (going from left to right) by the corresponding entry in Q (going from top to bottom) and then adding the results. ■

The Product Row \times Column

The **product** AB of a row matrix A and a column matrix B is a 1×1 matrix. The length of the row in A must match the length of the column in B for the product to be defined. To find the product, multiply each entry in A (going from left to right) by the corresponding entry in B (going from top to bottom) and then add the results.

Visualizing

	$2 \times 2 = 4$	Product of first entries = 4
	$4 \times 10 = 40$	Product of second entries = 40
	$1 \times (-1) = -1$	Product of third entries = -1
	$\underline{43}$	Sum of products = 43

quick Examples

$$1. [2 \quad 1] \begin{bmatrix} -3 \\ 1 \end{bmatrix} = [2 \times (-3) + 1 \times 1] = [-6 + 1] = [-5]$$

$$2. [2 \quad 4 \quad 1] \begin{bmatrix} 2 \\ 10 \\ -1 \end{bmatrix} = [2 \times 2 + 4 \times 10 + 1 \times (-1)] = [4 + 40 + (-1)] = [43]$$

Notes

1. In the discussion so far, *the row is on the left and the column is on the right* (RC again). (Later we will consider products where the column matrix is on the left and the row matrix is on the right.)
2. The row size has to match the column size. This means that, if we have a 1×3 row on the left, then the column on the right must be 3×1 in order for the product to make sense. For example, the product

$$[a \quad b \quad c] \begin{bmatrix} x \\ y \end{bmatrix}$$

is not defined. ■

Example 1 Revenue

The A-Plus auto parts store mentioned in examples in the previous section had the following sales in its Vancouver store:

	Vancouver
Wiper Blades	20
Cleaning Fluid (bottles)	10
Floor Mats	8

The store sells wiper blades for \$7.00 each, cleaning fluid for \$3.00 per bottle, and floor mats for \$12.00 each. Use matrix multiplication to find the total revenue generated by sales of these items.

Solution We need to multiply each sales figure by the corresponding price and then add the resulting revenue figures. We represent the sales by a column vector, as suggested by the table.

$$Q = \begin{bmatrix} 20 \\ 10 \\ 8 \end{bmatrix}$$

We put the selling prices in a row vector.

$$P = [7.00 \quad 3.00 \quad 12.00]$$

We can now compute the total revenue as the product

$$\begin{aligned} R = PQ &= [7.00 \quad 3.00 \quad 12.00] \begin{bmatrix} 20 \\ 10 \\ 8 \end{bmatrix} \\ &= [140.00 + 30.00 + 96.00] = [266.00] \end{aligned}$$

So, the sale of these items generated a total revenue of \$266.00.

Note We could also have written the quantity sold as a row vector (which would be Q^T) and the prices as a column vector (which would be P^T) and then multiplied them in the opposite order ($Q^T P^T$). Try this. ■

**using Technology**

See the Technology Guides at the end of the chapter to see how to multiply matrices using a TI-83/84 or Excel. To use the online Matrix Algebra Tool at

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enter the matrices P and Q as follows:

$$P = [7, 3, 12]$$

$$Q = [20$$

10

8]

Then type $P*Q$ in the formula box and press "Compute."

Example 2 Relationship with Linear Equations

a. Represent the matrix equation

$$\begin{bmatrix} 2 & -4 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = [5]$$

as an ordinary equation.

b. Represent the linear equation $3x + y - z + 2w = 8$ as a matrix equation.

Solution

a. If we perform the multiplication on the left, we get the 1×1 matrix $[2x - 4y + z]$. Thus the equation may be rewritten as

$$[2x - 4y + z] = [5]$$

Saying that these two 1×1 matrices are equal means that their entries are equal, so we get the equation

$$2x - 4y + z = 5$$

b. This is the reverse of part (a):

$$\begin{bmatrix} 3 & 1 & -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = [8]$$

† *Before we go on...* The row matrix $[3 \ 1 \ -1 \ 2]$ in Example 2 is the row of **coefficients** of the original equation (see Section 2.1). ■

Now to the general case of matrix multiplication:

The Product of Two Matrices: General Case

In general for matrices A and B , we can take the product AB only if the number of columns of A equals the number of rows of B (so that we can multiply the rows of A by the columns of B as above). The product AB is then obtained by taking its ij th entry to be:

$$ij\text{th entry of } AB = \text{Row } i \text{ of } A \times \text{Column } j \text{ of } B \quad \text{As defined above}$$

quick Examples

(R stands for row; C stands for column)

$$1. R_1 \rightarrow [2 \ 0 \ -1 \ 3] \begin{matrix} C_1 & C_2 & C_3 \\ \downarrow & \downarrow & \downarrow \\ \begin{bmatrix} 1 & 1 & -8 \\ 1 & -6 & 0 \\ 0 & 5 & 2 \\ -3 & 8 & 1 \end{bmatrix} \end{matrix} = [R_1 \times C_1 \ R_1 \times C_2 \ R_1 \times C_3] \\ = [-7 \ 21 \ -15]$$

$$2. \begin{matrix} R_1 \rightarrow [1 & -1] \\ R_2 \rightarrow [0 & 2] \end{matrix} \begin{matrix} C_1 & C_2 \\ \downarrow & \downarrow \\ \begin{bmatrix} 3 & 0 \\ 5 & -1 \end{bmatrix} \end{matrix} = \begin{bmatrix} R_1 \times C_1 & R_1 \times C_2 \\ R_2 \times C_1 & R_2 \times C_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 10 & -2 \end{bmatrix}$$

In matrix multiplication we always take

Rows on the left \times Columns on the right

Look at the dimensions in the two Quick Examples above.

$$\begin{array}{ccc} \text{Match} & & \text{Match} \\ \downarrow \downarrow & & \downarrow \downarrow \\ (1 \times 4)(4 \times 3) \rightarrow 1 \times 3 & & (2 \times 2)(2 \times 2) \rightarrow 2 \times 2 \end{array}$$

The fact that the number of columns in the left-hand matrix equals the number of rows in the right-hand matrix amounts to saying that the middle two numbers must match as above. If we “cancel” the middle matching numbers, we are left with the dimensions of the product.

Before continuing with examples, we state the rule for matrix multiplication formally.

Multiplication of Matrices: Formal Definition

If A is an $m \times n$ matrix and B is an $n \times k$ matrix, then the product AB is the $m \times k$ matrix whose ij th entry is the product

$$(AB)_{ij} = [a_{i1} \ a_{i2} \ a_{i3} \ \dots \ a_{in}] \begin{array}{c} \text{Row } i \text{ of } A \times \text{Column } j \text{ of } B \\ \downarrow \\ \begin{bmatrix} b_{1j} \\ b_{2j} \\ b_{3j} \\ \vdots \\ b_{nj} \end{bmatrix} \end{array} = a_{i1}b_{1j} + a_{i2}b_{2j} + a_{i3}b_{3j} + \dots + a_{in}b_{nj}$$

Example 3 Matrix Product

Calculate:

$$\mathbf{a.} \begin{bmatrix} 2 & 0 & -1 & 3 \\ 1 & -1 & 2 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 & -8 \\ 1 & 0 & 0 \\ 0 & 5 & 2 \\ -2 & 8 & -1 \end{bmatrix} \quad \mathbf{b.} \begin{bmatrix} -3 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & 1 \end{bmatrix}$$

Solution

a. Before we start the calculation, we check that the dimensions of the matrices match up.

$$\begin{array}{ccc} & \text{Match} & \\ & \swarrow \searrow & \\ 2 \times 4 & & 4 \times 3 \\ \begin{bmatrix} 2 & 0 & -1 & 3 \\ 1 & -1 & 2 & -2 \end{bmatrix} & & \begin{bmatrix} 1 & 1 & -8 \\ 1 & 0 & 0 \\ 0 & 5 & 2 \\ -2 & 8 & -1 \end{bmatrix} \end{array}$$

The product of the two matrices is defined, and the product will be a 2×3 matrix (we remove the matching 4s: $(2 \times 4)(4 \times 3) \rightarrow 2 \times 3$). To calculate the product, we

**using Technology**

See the Technology Guides at the end of the chapter for more on multiplying matrices using a TI-83/84 or Excel. To use the online Matrix Algebra Tool at

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enter the matrices A and B as follows:

$$A = \begin{bmatrix} 2, & 0, & -1, & 3 \\ 1, & -1, & 2, & -2 \end{bmatrix}$$

$$B = \begin{bmatrix} 1, & 1, & -8 \\ 1, & 0, & 0 \\ 0, & 5, & 2 \\ -2, & 8, & -1 \end{bmatrix}$$

Then type $A*B$ in the formula box and press “Compute.” If you try to multiply two matrices whose product is not defined, you will get an error alert box telling you that.

follow the prescription above:

$$\begin{array}{c} R_1 \rightarrow \\ R_2 \rightarrow \end{array} \begin{bmatrix} 2 & 0 & -1 & 3 \\ 1 & -1 & 2 & -2 \end{bmatrix} \begin{array}{c} C_1 \quad C_2 \quad C_3 \\ \downarrow \quad \downarrow \quad \downarrow \\ \begin{bmatrix} 1 & 1 & -8 \\ 1 & 0 & 0 \\ 0 & 5 & 2 \\ -2 & 8 & -1 \end{bmatrix} \end{array} = \begin{bmatrix} R_1 \times C_1 & R_1 \times C_2 & R_1 \times C_3 \\ R_2 \times C_1 & R_2 \times C_2 & R_2 \times C_3 \end{bmatrix} \\ = \begin{bmatrix} -4 & 21 & -21 \\ 4 & -5 & -2 \end{bmatrix}$$

b. The dimensions of the two matrices given are 2×1 and 1×2 . Because the 1s match, the product is defined, and the result will be a 2×2 matrix.

$$\begin{array}{c} R_1 \rightarrow \\ R_2 \rightarrow \end{array} \begin{bmatrix} -3 \\ 1 \end{bmatrix} \begin{array}{c} C_1 \quad C_2 \\ \downarrow \quad \downarrow \\ \begin{bmatrix} 2 \\ 1 \end{bmatrix} \end{array} = \begin{bmatrix} R_1 \times C_1 & R_1 \times C_2 \\ R_2 \times C_1 & R_2 \times C_2 \end{bmatrix} = \begin{bmatrix} -6 & -3 \\ 2 & 1 \end{bmatrix}$$

Note In part (a) we *cannot* multiply the matrices in the opposite order—the dimensions do not match. We say simply that the product in the opposite order is **not defined**. In part (b) we *can* multiply the matrices in the opposite order, but we would get a 1×1 matrix if we did so. Thus, order is important when multiplying matrices. In general, if AB is defined, then BA need not even be defined. If BA is also defined, it may not have the same dimensions as AB . And even if AB and BA have the same dimensions, they may have different entries (see the next example). ■

Example 4 AB versus BA

Let $A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 3 & 0 \\ 5 & -1 \end{bmatrix}$. Find AB and BA .

Solution Note first that A and B are both 2×2 matrices, so the products AB and BA are both defined and are both 2×2 matrices—unlike the case in Example 3(b). We first calculate AB :

$$AB = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 5 & -1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ 10 & -2 \end{bmatrix}$$

Now let's calculate BA .

$$BA = \begin{bmatrix} 3 & 0 \\ 5 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ 5 & -7 \end{bmatrix}$$

Notice that BA has no resemblance to AB ! Thus, we have discovered that, even for square matrices:

Matrix multiplication is not commutative.

In other words, $AB \neq BA$ in general, even when AB and BA both exist and have the same dimensions. (There are instances when $AB = BA$ for particular matrices A and B , but this is an exception, not the rule.)

Example 5 Revenue

January sales at the A-Plus auto parts stores in Vancouver and Quebec are given in the following table.

	Vancouver	Quebec
Wiper Blades	20	15
Cleaning Fluid (bottles)	10	12
Floor Mats	8	4

The usual selling prices for these items are \$7.00 each for wiper blades, \$3.00 per bottle for cleaning fluid, and \$12.00 each for floor mats. The discount prices for A-Plus Club members are \$6.00 each for wiper blades, \$2.00 per bottle for cleaning fluid, and \$10.00 each for floor mats. Use matrix multiplication to compute the total revenue at each store, assuming first that all items were sold at the usual prices, and then that they were all sold at the discount prices.

Solution We can do all of the requested calculations at once with a single matrix multiplication. Consider the following two labeled matrices.

$$Q = \begin{matrix} & \mathbf{V} & \mathbf{Q} \\ \mathbf{Wb} & \begin{bmatrix} 20 & 15 \end{bmatrix} \\ \mathbf{Cf} & \begin{bmatrix} 10 & 12 \end{bmatrix} \\ \mathbf{Fm} & \begin{bmatrix} 8 & 4 \end{bmatrix} \end{matrix}$$

$$P = \begin{matrix} & \mathbf{Wb} & \mathbf{Cf} & \mathbf{Fm} \\ \mathbf{Usual} & \begin{bmatrix} 7.00 & 3.00 & 12.00 \end{bmatrix} \\ \mathbf{Discount} & \begin{bmatrix} 6.00 & 2.00 & 10.00 \end{bmatrix} \end{matrix}$$

The first matrix records the quantities sold, while the second records the sales prices under the two assumptions. To compute the revenue at both stores under the two different assumptions, we calculate $R = PQ$.

$$\begin{aligned} R = PQ &= \begin{bmatrix} 7.00 & 3.00 & 12.00 \\ 6.00 & 2.00 & 10.00 \end{bmatrix} \begin{bmatrix} 20 & 15 \\ 10 & 12 \\ 8 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 266.00 & 189.00 \\ 220.00 & 154.00 \end{bmatrix} \end{aligned}$$

We can label this matrix as follows.

$$R = \begin{matrix} & \mathbf{V} & \mathbf{Q} \\ \mathbf{Usual} & \begin{bmatrix} 266.00 & 189.00 \end{bmatrix} \\ \mathbf{Discount} & \begin{bmatrix} 220.00 & 154.00 \end{bmatrix} \end{matrix}$$

In other words, if the items were sold at the usual price, then Vancouver had a revenue of \$266 while Quebec had a revenue of \$189, and so on.

† *Before we go on...* In Example 5 we were able to calculate PQ because the dimensions matched correctly: $(2 \times 3)(3 \times 2) \rightarrow 2 \times 2$. We could also have multiplied them in the opposite order and gotten a 3×3 matrix. Would the product QP be meaningful? In an

application like this, not only do the dimensions have to match, but also the *labels* have to match for the result to be meaningful. The labels on the three columns of P are the parts that were sold, and these are also the labels on the three rows of Q . Therefore, we can “cancel labels” at the same time that we cancel the dimensions in the product. However, the labels on the two columns of Q do not match the labels on the two rows of P , and there is no useful interpretation of the product QP in this situation. ■

There are very special square matrices of every size: 1×1 , 2×2 , 3×3 , and so on, called the **identity** matrices.

Identity Matrix

The $n \times n$ identity matrix I is the matrix with 1s down the **main diagonal** (the diagonal starting at the top left) and 0s everywhere else. In symbols,

$$I_{ii} = 1, \quad \text{and} \\ I_{ij} = 0 \quad \text{if } i \neq j$$

quick Examples

1. 1×1 identity matrix $I = [1]$

2. 2×2 identity matrix $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

3. 3×3 identity matrix $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

4. 4×4 identity matrix $I = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Note Identity matrices are always square matrices, meaning that they have the same number of rows as columns. There is no such thing, for example, as the “ 2×4 identity matrix.”

The next example shows why I is interesting. ■

Example 6 Identity Matrix

Evaluate the products AI and IA , where $A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$ and I is the 3×3 identity matrix.

Solution

First notice that A is arbitrary; it could be any 3×3 matrix.

$$AI = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

**using Technology**

See the Technology Guides at the end of the chapter to see how to get an identity matrix using a TI-83/84 or Excel. In the online Matrix Algebra Tool at

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use I in a formula to refer to the identity matrix of any dimension. The program will choose the correct dimension in the context of the formula. For example, if A is a 3×3 matrix, then the expression $I-A$ uses the 3×3 identity matrix for I .

and

$$IA = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

In both cases, the answer is the matrix A we started with. In symbols,

$$AI = A$$

and

$$IA = A$$

no matter which 3×3 matrix A you start with. Now this should remind you of a familiar fact from arithmetic:

$$a \cdot 1 = a$$

and

$$1 \cdot a = a$$

That is why we call the matrix I the 3×3 *identity* matrix, because it appears to play the same role for 3×3 matrices that the identity 1 does for numbers.

✦ *Before we go on...* Try a similar calculation using 2×2 matrices: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

let I be the 2×2 identity matrix, and check that $AI = IA = A$. In fact, the equation

$$AI = IA = A$$

works for square matrices of every dimension. It is also interesting to notice that $AI = A$ if I is the 2×2 identity matrix and A is any 3×2 matrix (try one). In fact, if I is any identity matrix, then $AI = A$ whenever the product is defined, and $IA = A$ whenever this product is defined. ■

We can now add to the list of properties we gave for matrix arithmetic at the end of Section 3.1 by writing down properties of matrix multiplication. In stating these properties, we shall assume that all matrix products we write are defined—that is, that the matrices have correctly matching dimensions. The first eight properties are the ones we've already seen; the rest are new.

Properties of Matrix Addition and Multiplication

If A , B and C are matrices, if O is a zero matrix, and if I is an identity matrix, then the following hold:

$A + (B + C) = (A + B) + C$	<i>Additive associative law</i>
$A + B = B + A$	<i>Additive commutative law</i>
$A + O = O + A = A$	<i>Additive identity law</i>
$A + (-A) = O = (-A) + A$	<i>Additive inverse law</i>
$c(A + B) = cA + cB$	<i>Distributive law</i>
$(c + d)A = cA + dA$	<i>Distributive law</i>

$1A = A$	<i>Scalar unit</i>
$0A = O$	<i>Scalar zero</i>
$A(BC) = (AB)C$	<i>Multiplicative associative law</i>
$c(AB) = (cA)B$	<i>Multiplicative associative law</i>
$c(dA) = (cd)A$	<i>Multiplicative associative law</i>
$AI = IA = A$	<i>Multiplicative identity law</i>
$A(B + C) = AB + AC$	<i>Distributive law</i>
$(A + B)C = AC + BC$	<i>Distributive law</i>
$OA = AO = O$	<i>Multiplication by zero matrix</i>

Note that we have not included a multiplicative commutative law for matrices, because the equation $AB = BA$ does not hold in general. In other words, matrix multiplication is *not* exactly like multiplication of numbers. (You have to be a little careful because it is easy to apply the commutative law without realizing it.)

We should also say a bit more about transposition. Transposition and multiplication have an interesting relationship. We write down the properties of transposition again, adding one new one.

Properties of Transposition

$$\begin{aligned}(A + B)^T &= A^T + B^T \\ (cA)^T &= c(A^T) \\ (AB)^T &= B^T A^T\end{aligned}$$

Notice the change in order in the last one. The order is crucial.

quick Examples

- $\left(\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 5 & -1 \end{bmatrix}\right)^T = \begin{bmatrix} -2 & 1 \\ 10 & -2 \end{bmatrix}^T = \begin{bmatrix} -2 & 10 \\ 1 & -2 \end{bmatrix} \quad (AB)^T$
- $\begin{bmatrix} 3 & 0 \\ 5 & -1 \end{bmatrix}^T \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}^T = \begin{bmatrix} 3 & 5 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} -2 & 10 \\ 1 & -2 \end{bmatrix} \quad B^T A^T$
- $\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}^T \begin{bmatrix} 3 & 0 \\ 5 & -1 \end{bmatrix}^T = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3 & 5 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ -3 & -7 \end{bmatrix} \quad A^T B^T$

These properties give you a glimpse of the field of mathematics known as **abstract algebra**. Algebraists study operations like these that resemble the operations on numbers but differ in some way, such as the lack of commutativity for multiplication seen here.

We end this section with more on the relationship between linear equations and matrix equations, which is one of the important applications of matrix multiplication.

Example 7 Matrix Form of a System of Linear Equations

a. If

$$A = \begin{bmatrix} 1 & -2 & 3 \\ 2 & 0 & -1 \\ -3 & 1 & 1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \text{ and } B = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}$$

rewrite the matrix equation $AX = B$ as a system of linear equations.

b. Express the following system of equations as a matrix equation of the form $AX = B$:

$$2x + y = 3$$

$$4x - y = -1$$

Solution

a. The matrix equation $AX = B$ is

$$\begin{bmatrix} 1 & -2 & 3 \\ 2 & 0 & -1 \\ -3 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}$$

As in Example 2(a), we first evaluate the left-hand side and then set it equal to the right-hand side.

$$\begin{bmatrix} 1 & -2 & 3 \\ 2 & 0 & -1 \\ -3 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x - 2y + 3z \\ 2x - z \\ -3x + y + z \end{bmatrix}$$

$$\begin{bmatrix} x - 2y + 3z \\ 2x - z \\ -3x + y + z \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}$$

Because these two matrices are equal, their corresponding entries must be equal.

$$x - 2y + 3z = 3$$

$$2x - z = -1$$

$$-3x + y + z = 0$$

In other words, the matrix equation $AX = B$ is equivalent to this system of linear equations. Notice that the coefficients of the left-hand sides of these equations are the entries of the matrix A . We call A the **coefficient matrix** of the system of equations. The entries of X are the unknowns and the entries of B are the right-hand sides 3, -1, and 0.

b. As we saw in part (a), the coefficient matrix A has entries equal to the coefficients of the left-hand sides of the equations. Thus,

$$A = \begin{bmatrix} 2 & 1 \\ 4 & -1 \end{bmatrix}$$

X is the column matrix consisting of the unknowns, while B is the column matrix consisting of the right-hand sides of the equations, so

$$X = \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

The system of equations can be rewritten as the matrix equation $AX = B$ with this A , X , and B .

This translation of systems of linear equations into matrix equations is really the first step in the method of solving linear equations discussed in Chapter 2. There we worked with the **augmented matrix** of the system, which is simply A with B adjoined as an extra column.

Q: When we write a system of equations as $AX = B$, couldn't we solve for the unknown X by dividing both sides by A ?

A: If we interpret division as multiplication by the inverse (for example, $2 \div 3 = 2 \times 3^{-1}$), we shall see in the next section that *certain* systems of the form $AX = B$ can be solved in this way, by multiplying both sides by A^{-1} . We first need to discuss what we mean by A^{-1} and how to calculate it. ■

3.2 EXERCISES

● denotes basic skills exercises

◆ denotes challenging exercises

techEx indicates exercises that should be solved using technology

In Exercises 1–28, compute the products. Some of these may be undefined. Exercises marked **tech**Ex should be done using technology. The others should be done two ways: by hand and by using technology where possible. *hint* [see Example 3]

$$1. \bullet [1 \ 3 \ -1] \begin{bmatrix} 9 \\ 1 \\ -1 \end{bmatrix}$$

$$2. \bullet [4 \ 0 \ -1] \begin{bmatrix} -4 \\ 1 \\ 8 \end{bmatrix}$$

$$3. \bullet \left[-1 \ \frac{1}{2}\right] \begin{bmatrix} -\frac{1}{3} \\ 1 \end{bmatrix}$$

$$4. \bullet [-1 \ 1] \begin{bmatrix} \frac{3}{4} \\ \frac{1}{4} \end{bmatrix}$$

$$5. \bullet [0 \ -2 \ 1] \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$6. \bullet [4 \ -1 \ 1] \begin{bmatrix} -x \\ x \\ y \end{bmatrix}$$

$$7. \bullet [1 \ 3 \ 2] \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$8. \bullet [3 \ 2][1 \ -2]$$

$$9. \bullet [-1 \ 1] \begin{bmatrix} -3 & 1 & 4 & 3 \\ 0 & 1 & -2 & 1 \end{bmatrix}$$

$$10. \bullet [2, \ -1] \begin{bmatrix} -3 & 1 & 4 & 3 \\ 4 & 0 & 1 & 3 \end{bmatrix}$$

$$11. \bullet [1 \ -1 \ 2 \ 3] \begin{bmatrix} -1 & 2 & 0 \\ 2 & -1 & 0 \\ 0 & 5 & 2 \\ -1 & 8 & 1 \end{bmatrix}$$

$$12. \bullet [0 \ 1 \ -1 \ 2] \begin{bmatrix} 1 & -2 & 1 \\ 0 & 1 & 3 \\ 6 & 0 & 0 \\ -1 & -2 & 11 \end{bmatrix}$$

$$13. \bullet \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ 4 & 8 & 0 \end{bmatrix}$$

$$14. \bullet \begin{bmatrix} 0 & 1 & -1 \\ 3 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 4 & 2 \\ 0 & 1 \end{bmatrix}$$

$$15. \bullet \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

$$16. \bullet \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 3 & -3 \\ 5 & -7 \end{bmatrix}$$

$$17. \bullet \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$$

$$18. \bullet \begin{bmatrix} 3 & -3 \\ 5 & -7 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}$$

$$19. \bullet \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 2 & 3 \end{bmatrix}$$

$$20. \bullet \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & -3 \\ 2 & -1 \end{bmatrix}$$

$$21. \bullet \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 2 & 3 \\ 1 & 1 \end{bmatrix}$$

$$22. \bullet \begin{bmatrix} 0 & 1 & -1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & -3 \\ 2 & -1 \end{bmatrix}$$

$$23. \bullet \begin{bmatrix} 1 & 0 & -1 \\ 2 & -2 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 4 \\ 1 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}$$

$$24. \bullet \begin{bmatrix} 1 & 2 & 0 \\ 4 & -1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -4 \\ 4 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$25. \bullet \begin{bmatrix} 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 1 \\ -2 & 0 & 1 & 4 \\ 0 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \\ 2 \\ 0 \end{bmatrix}$$

$$26. \bullet \begin{bmatrix} 1 & 1 & -7 & 0 \\ -1 & 0 & 2 & 4 \\ -1 & 0 & -2 & 1 \\ 1 & -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \\ 2 \\ 1 \end{bmatrix}$$

$$27. \text{tech Ex } \begin{bmatrix} 1.1 & 2.3 & 3.4 & -1.2 \\ 3.4 & 4.4 & 2.3 & 1.1 \\ 2.3 & 0 & -2.2 & 1.1 \\ 1.2 & 1.3 & 1.1 & 1.1 \end{bmatrix} \begin{bmatrix} -2.1 & 0 & -3.3 \\ -3.4 & -4.8 & -4.2 \\ 3.4 & 5.6 & 1 \\ 1 & 2.2 & 9.8 \end{bmatrix}$$

● basic skills ◆ challenging **tech**Ex technology exercise

$$28. \text{tech Ex } \begin{bmatrix} 1.2 & 2.3 & 3.4 & 4.5 \\ 3.3 & 4.4 & 5.5 & 6.6 \\ 2.3 & -4.3 & -2.2 & 1.1 \\ 2.2 & -1.2 & -1 & 1.1 \end{bmatrix} \begin{bmatrix} 9.8 & 1 & -1.1 \\ 8.8 & 2 & -2.2 \\ 7.7 & 3 & -3.3 \\ 6.6 & 4 & -4.4 \end{bmatrix}$$

29. ● Find* $A^2 = A \cdot A$, $A^3 = A \cdot A \cdot A$, A^4 , and A^{100} , given that

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

30. ● Repeat the preceding exercise with $A = \begin{bmatrix} 0 & 2 & 0 & -1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$.

Exercises 31–38 should be done two ways: (a) by hand; (b) using technology where possible.

Let

$$A = \begin{bmatrix} 0 & -1 & 0 & 1 \\ 10 & 0 & 1 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & -1 \\ 1 & 1 \\ -1 & 3 \\ 5 & 0 \end{bmatrix}, C = \begin{bmatrix} 1 & -1 \\ 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Evaluate:

31. ● AB 32. ● AC 33. ● $A(B - C)$ 34. ● $(B - C)A$

Let

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 0 & -2 \end{bmatrix}, B = \begin{bmatrix} 3 & 0 & -1 \\ 5 & -1 & 1 \end{bmatrix}, C = \begin{bmatrix} x & 1 & w \\ z & r & 4 \end{bmatrix}$$

Evaluate:

35. ● AB 36. ● AC 37. ● $A(B + C)$ 38. ● $(B + C)A$

In Exercises 39–44, calculate (a) $P^2 = P \cdot P$ (b) $P^4 = P^2 \cdot P^2$ and (c) P^8 . (Round all entries to four decimal places.)
(d) Without computing it explicitly, find P^{100} .

$$39. P = \begin{bmatrix} 0.2 & 0.8 \\ 0.2 & 0.8 \end{bmatrix} \quad 40. P = \begin{bmatrix} 0.1 & 0.9 \\ 0.1 & 0.9 \end{bmatrix}$$

$$41. P = \begin{bmatrix} 0.1 & 0.9 \\ 0 & 1 \end{bmatrix} \quad 42. P = \begin{bmatrix} 1 & 0 \\ 0.8 & 0.2 \end{bmatrix}$$

$$43. P = \begin{bmatrix} 0.25 & 0.25 & 0.50 \\ 0.25 & 0.25 & 0.50 \\ 0.25 & 0.25 & 0.50 \end{bmatrix}$$

$$44. P = \begin{bmatrix} 0.3 & 0.3 & 0.4 \\ 0.3 & 0.3 & 0.4 \\ 0.3 & 0.3 & 0.4 \end{bmatrix}$$

* $A \cdot A \cdot A$ is $A(A \cdot A)$, or the equivalent $(A \cdot A)A$ by the associative law. Similarly, $A \cdot A \cdot A \cdot A = A(A \cdot A \cdot A) = (A \cdot A \cdot A)A = (A \cdot A)(A \cdot A)$; it doesn't matter where we place parentheses.

In Exercises 45–48, translate the given matrix equations into systems of linear equations. *hint* [see Example 7]

$$45. \bullet \begin{bmatrix} 2 & -1 & 4 \\ -4 & \frac{3}{4} & \frac{1}{3} \\ -3 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}$$

$$46. \bullet \begin{bmatrix} 1 & -1 & 4 \\ -\frac{1}{3} & -3 & \frac{1}{3} \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -3 \\ -1 \\ 2 \end{bmatrix}$$

$$47. \bullet \begin{bmatrix} 1 & -1 & 0 & 1 \\ 1 & 1 & 2 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

$$48. \bullet \begin{bmatrix} 0 & 1 & 6 & 1 \\ 1 & -5 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} -2 \\ 9 \end{bmatrix}$$

In Exercises 49–52, translate the given systems of equations into matrix form. *hint* [see Example 7]

$$49. \bullet \begin{cases} x - y = 4 \\ 2x - y = 0 \end{cases} \quad 50. \bullet \begin{cases} 2x + y = 7 \\ -x = 9 \end{cases}$$

$$51. \bullet \begin{cases} x + y - z = 8 \\ 2x + y + z = 4 \end{cases} \quad 52. \bullet \begin{cases} x + y + 2z = -2 \\ 4x + 2y - z = -8 \end{cases}$$

$$\frac{3x}{4} + \frac{z}{2} = 1 \quad \frac{x}{2} + \frac{y}{3} = 1$$

Applications

53. ● **Revenue** Your T-shirt operation is doing a booming trade. Last week you sold 50 tie-dye shirts for \$15 each, 40 Suburban State University Crew shirts for \$10 each, and 30 Lacrosse T-shirts for \$12 each. Use matrix operations to calculate your total revenue for the week. *hint* [see Example 1]

54. ● **Revenue** Karen Sandberg, your competitor in Suburban State U's T-shirt market, has apparently been undercutting your prices and outperforming you in sales. Last week she sold 100 tie dye shirts for \$10 each, 50 (low quality) Crew shirts at \$5 apiece, and 70 Lacrosse T-shirts for \$8 each. Use matrix operations to calculate her total revenue for the week.

55. ● **Revenue** Recall the Left Coast Bookstore chain from the preceding section. In January, it sold 700 hardcover books, 1300 softcover books, and 2000 plastic books in San Francisco; it sold 400 hardcover, 300 softcover, and 500 plastic books in Los Angeles. Now, hardcover books sell for \$30 each, softcover books sell for \$10 each, and plastic books sell for \$15 each. Write a column matrix with the price data and show how matrix multiplication (using the sales and price data matrices) may be used to compute the total revenue at the two stores. *hint* [see Example 5]

56. ● **Profit** Refer back to Exercise 55, and now suppose that each hardcover book costs the stores \$10, each softcover book

costs \$5, and each plastic book costs \$10. Use matrix operations to compute the total *profit* at each store in January.

57. ● **Publishing** Editors' workloads were increasing during the 1990's, as the following table shows.⁷

	1993	1994	1995	1996
Books per Editor	3	3.5	5	5.2
Number of Editors	16,000	15,000	12,500	13,000

Use matrix multiplication to estimate the total number of books published during the years 1993–1996.

58. ● **Real Estate** The following table shows the cost of one square foot of residential real estate, in dollars per square foot, at the start of 2003⁸ together with the number of square feet your development company intends to purchase in each city.

	New York	London	Hong Kong
Cost per sq. foot	760	800	300
Number of sq. ft	500	800	1000

Use matrix multiplication to estimate the total cost of the real estate.

tech Ex **Income** Exercises 59–62 are based on the following Excel sheet, which shows the 2004 and 2005 male and female population in various age groups, as well as per capita incomes.⁹

	A	B	C		D		E		F	G
	Age	Mean Income (\$1000)	2004 Population (Millions)		2005 Population (Millions)					
			Female	Male	Female	Male	Female	Male		
4	15 to 24	12.5	20	21	20	21				
5	25 to 34	31.7	19	18	19	18				
6	35 to 44	40.4	22	21	22	21				
7	45 to 54	43.0	21	21	21	20				
8	55 to 64	38.0	16	14	15	14				
9	65 to 74	25.2	42	42	42	42				

59. ● Use matrix algebra to estimate the total income for females in 2004. (Round the answer to two significant digits.)
60. ● Use matrix algebra to estimate the total income for males in 2005.

⁷ Books per Editor refers to new titles. Data are rounded. SOURCES: R.R. Bowker, EEOC/*The New York Times*, June 29, 1998, p. E1.

⁸ Figures are rounded. SOURCE: CLSA, Honk Kong Monetary Authority/*New York Times*, August 15, 2003, p. C1.

⁹ The population figures are Census Bureau estimates, and the income figures are 2001 mean per capita incomes. All figures are rounded. SOURCE: U.S. Census Bureau www.census.gov.

61. ● Give a single matrix formula that expresses the difference in total income between males and females in 2005, and compute its value, rounded to two significant digits.
62. ● Give a single matrix formula that expresses the total income in 2005, and compute its value, rounded to two significant digits.
63. ● **Cheese Production** The total amount of cheese, in billions of pounds, produced in the 13 western and 12 north central U.S. states in 1999 and 2000 was as follows.¹⁰

	1999	2000
Western States	2.7	3.0
North Central States	3.9	4.0

Thinking of this table as a (labeled) 2×2 matrix P , compute the matrix product $[-1 \ 1]P$. What does this product represent?

64. ● **Milk Production** The total amount of milk, in billions of pounds, produced in the 13 western and 12 north central U.S. states in 1999 and 2000 was as follows.¹¹

	1999	2000
Western States	56	60
North Central States	57	59

Thinking of this table as a (labeled) 2×2 matrix P , compute the matrix product $P \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. What does this product represent?

Bankruptcy Filings Exercises 65–70 are based on the following chart, which shows the number of personal bankruptcy filings in three New York/New Jersey regions during various months of 2001–2002.¹²

	Jan 01	Jul 01	Jan 02
Manhattan	150	150	150
Brooklyn	300	300	250
Newark	250	250	200

65. Each month, your law firm handles 10% of all bankruptcy filings in Manhattan, 5% of all filings in Brooklyn, and 20% of all filings in Newark. Use matrix multiplication to compute the total number of bankruptcy filings handled by your firm in each of the months shown.

¹⁰ Ibid.

¹¹ Figures are approximate. SOURCE: Department of Agriculture/*New York Times*, June 28, 2001, p. C1.

¹² Data are approximate four-week moving averages. SOURCE: Lundquist Consulting/*New York Times*, February 10, 2002, p. L1.

66. Your law firm handled 10% of all bankruptcy filings in each region in January 2001, 30% of all filings in July 2001, and 20% of all filings in January 2002. Use matrix multiplication to compute the total number of bankruptcy filings handled by your firm in each of the regions shown.
67. Let A be the 3×3 matrix whose entries are the figures in the table, and let $B = [1 \ 1 \ 0]$. What does the matrix BA represent?
68. Let A be the 3×3 matrix whose entries are the figures in the table, and let $B = [1 \ 1 \ 0]^T$. What does the matrix AB represent?
69. Write a matrix product whose computation gives the total number by which the combined filings in Manhattan and Newark exceeded the filings in Brooklyn.
70. Write a matrix product whose computation gives the total number by which bankruptcy filings in January, 2001, exceeded filings in January, 2002.
71. **Costs** Microbucks Computer Co. makes two computers, the Pomegranate II and the Pomegranate Classic. The Pom II requires 2 processor chips, 16 memory chips, and 20 vacuum tubes, while the Pom Classic requires 1 processor chip, 4 memory chips, and 40 vacuum tubes. There are two companies that can supply these parts: Motorel can supply them at \$100 per processor chip, \$50 per memory chip, and \$10 per vacuum tube, while Intola can supply them at \$150 per processor chip, \$40 per memory chip, and \$15 per vacuum tube. Write down all of this data in two matrices, one showing the parts required for each model computer, and the other showing the prices for each part from each supplier. Then show how matrix multiplication allows you to compute the total cost for parts for each model when parts are bought from either supplier.
72. **Profits** Refer back to Exercise 71. It actually costs Motorel only \$25 to make each processor chip, \$10 for each memory chip, and \$5 for each vacuum tube. It costs Intola \$50 per processor chip, \$10 per memory chip, and \$7 per vacuum tube. Use matrix operations to find the total profit Motorel and Intola would make on each model.
73. **Tourism** The following table gives the number of people (in thousands) who visited Australia and South Africa in 1998.¹³

		To	Australia	South Africa
From	North America		440	190
	Europe		950	950
	Asia		1790	200

¹³ Figures are rounded to the nearest 10,000. SOURCES: South African Dept. of Environmental Affairs and Tourism; Australia Tourist Commission/ *The New York Times*, January 15, 2000, p. C1.

You estimate that 5% of all visitors to Australia and 4% of all visitors to South Africa decide to settle there permanently. Take A to be the 3×2 matrix whose entries are the 1998 tourism figures in the above table and take

$$B = \begin{bmatrix} 0.05 \\ 0.04 \end{bmatrix} \quad \text{and} \quad C = \begin{bmatrix} 0.05 & 0 \\ 0 & 0.04 \end{bmatrix}$$

Compute the products AB and AC . What do the entries in these matrices represent?

74. ● **Tourism** Referring to the tourism figures in the preceding exercise, you estimate that from 1998 to 2008, tourism from North America to each of Australia and South Africa will have increased by 20%, tourism from Europe by 30%, and tourism from Asia by 10%. Take A to be the 3×2 matrix whose entries are the 1998 tourism figures and take

$$B = [1.2 \ 1.3 \ 1.1] \quad \text{and} \quad C = \begin{bmatrix} 1.2 & 0 & 0 \\ 0 & 1.3 & 0 \\ 0 & 0 & 1.1 \end{bmatrix}$$

Compute the products AB and AC . What do the entries in these matrices represent?

75. ♦ **tech**Ex **Population Movement** In 2003, the population of the U.S., broken down by regions, was 53.3 million in the Northeast, 64.0 million in the Midwest, 101.6 million in the South, and 65.4 million in the West.¹⁴ The matrix P below shows the population movement during the period 2003–2004. (Thus, 98.79% of the population in the Northeast stayed there, while 0.20% of the population in the Northeast moved to the Midwest, and so on.)

$$P = \begin{array}{l} \text{From NE} \\ \text{From MW} \\ \text{From S} \\ \text{From W} \end{array} \begin{array}{cccc} \text{To NE} & \text{To MW} & \text{To S} & \text{To W} \\ \begin{bmatrix} 0.9879 & 0.0020 & 0.0081 & 0.0019 \\ 0.0014 & 0.9895 & 0.0063 & 0.0028 \\ 0.0027 & 0.0025 & 0.9927 & 0.0022 \\ 0.0010 & 0.0030 & 0.0050 & 0.9909 \end{bmatrix} \end{array}$$

Set up the 2003 population figures as a row vector. Then use matrix multiplication to compute the population in each region in 2004. (Round all answers to the nearest 0.1 million.)

76. ♦ **tech**Ex **Population Movement** Assume that the percentages given in the preceding exercise also describe the population movements from 2004–2005. Use two matrix multiplications to predict from the data in the preceding exercise the population in each region in 2005. (Round the *final* answers to the nearest 0.1 million, but do not round the intermediate answer.)

¹⁴ Note that this exercise ignores migration into or out of the country. The internal migration figures and 2004 population figures are accurate. SOURCE: U.S. Census Bureau, Current Population Survey, 2004 Annual Social and Economic Supplement, Internet release date June 23, 2005. www.census.gov.

Communication and Reasoning Exercises

77. ● Give an example of two matrices A and B such that AB is defined but BA is not defined.
78. ● Give an example of two matrices A and B of different dimensions such that both AB and BA are defined.
79. ● Compare addition and multiplication of 1×1 matrices to the arithmetic of numbers.
80. ● In comparing the algebra of 1×1 matrices, as discussed so far, to the algebra of real numbers (see Exercise 79), what important difference do you find?
81. ● Comment on the following claim: Every matrix equation represents a system of equations.
82. ● When is it true that both AB and BA are defined, even though neither A nor B is a square matrix?
83. Find a scenario in which it would be useful to “multiply” two row vectors according to the rule

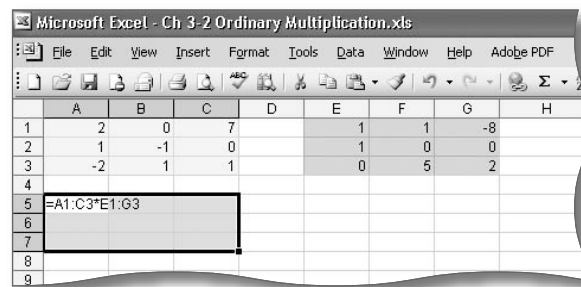
$$[a \ b \ c][d \ e \ f] = [ad \ be \ cf]$$

84. Make up an application whose solution reads as follows.

$$\text{“Total revenue} = [10 \ 100 \ 30] \begin{bmatrix} 10 & 0 & 3 \\ 1 & 2 & 0 \\ 0 & 1 & 40 \end{bmatrix}\text{”}$$

● basic skills ◆ challenging **tech** Ex technology exercise

85. What happens in Excel if, instead of using the function `MMULT`, you use “ordinary multiplication” as shown here?



86. Define the *naïve product* $A \square B$ of two $m \times n$ matrices A and B by

$$(A \square B)_{ij} = A_{ij} B_{ij}$$

(This is how someone who has never seen matrix multiplication before might think to multiply matrices.) Referring to Example 1 in this section, compute and comment on the meaning of $P \square (Q^T)$.

3.3 Matrix Inversion

Now that we’ve discussed matrix addition, subtraction, and multiplication, you may well be wondering about matrix *division*. In the realm of real numbers, division can be thought of as a form of multiplication: Dividing 3 by 7 is the same as multiplying 3 by $1/7$, the inverse of 7. In symbols, $3 \div 7 = 3 \times (1/7)$, or 3×7^{-1} . In order to imitate division of real numbers in the realm of matrices, we need to discuss the multiplicative **inverse**, A^{-1} , of a matrix A .

Note Because multiplication of real numbers is commutative, we can write, for example, $\frac{3}{7}$ as either 3×7^{-1} or $7^{-1} \times 3$. In the realm of matrices, multiplication is not commutative, so from now on we shall *never* talk about “division” of matrices (by “ $\frac{B}{A}$ ” should we mean $A^{-1}B$ or BA^{-1} ?). ■

Before we try to find the inverse of a matrix, we must first know exactly what we *mean* by the inverse. Recall that the inverse of a number a is the number, often written a^{-1} , with the property that $a^{-1} \cdot a = a \cdot a^{-1} = 1$. For example, the inverse of 76 is the number $76^{-1} = 1/76$, because $(1/76) \cdot 76 = 76 \cdot (1/76) = 1$. This is the number calculated by the x^{-1} button found on most calculators. Not all numbers have an inverse.

For example—and this is the only example—the number 0 has no inverse, because you cannot get 1 by multiplying 0 by anything.

The inverse of a matrix is defined similarly. To make life easier, we shall restrict attention to **square** matrices, matrices that have the same number of rows as columns.¹⁵

Inverse of a Matrix

The **inverse** of an $n \times n$ matrix A is that $n \times n$ matrix A^{-1} which, when multiplied by A on either side, yields the $n \times n$ identity matrix I . Thus,

$$AA^{-1} = A^{-1}A = I$$

If A has an inverse, it is said to be **invertible**. Otherwise, it is said to be **singular**.

quick Examples

1. The inverse of the 1×1 matrix $[3]$ is $[1/3]$, because $[3][1/3] = [1] = [1/3][3]$.
2. The inverse of the $n \times n$ identity matrix I is I itself, because $I \times I = I$. Thus, $I^{-1} = I$.
3. The inverse of the 2×2 matrix $A = \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}$ is $A^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$, because

$$\begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad AA^{-1} = I$$

$$\text{and} \quad \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad A^{-1}A = I$$

Notes

1. It is possible to show that if A and B are square matrices with $AB = I$, then it must also be true that $BA = I$. In other words, once we have checked that $AB = I$, we know that B is the inverse of A . The second check, that $BA = I$, is unnecessary.
2. If B is the inverse of A , then we can also say that A is the inverse of B (why?). Thus, we sometimes refer to such a pair of matrices as an **inverse pair** of matrices. ■

Example 1 Singular Matrix

Can $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ have an inverse?

Solution No. To see why not, notice that both entries in the second row of AB will be 0, no matter what B is. So AB cannot equal I , no matter what B is. Hence, A is singular.

† *Before we go on...* If you think about it, you can write down many similar examples of singular matrices. There is only one number with no multiplicative inverse (0), but there are many matrices having no inverses. ■

¹⁵Nonsquare matrices *cannot* have inverses in the sense that we are talking about here. This is not a trivial fact to prove.

Finding the Inverse of a Square Matrix

Q: In the box, it was stated that the inverse of $\begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}$ is $\begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$. How was that obtained?

A: We can think of the problem of finding A^{-1} as a problem of finding four unknowns, the four unknown entries of A^{-1} :

$$A^{-1} = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$$

These unknowns must satisfy the equation $AA^{-1} = I$, or

$$\begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x & y \\ z & w \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

If we were to try to find the first column of A^{-1} , consisting of x and z , we would have to solve

$$\begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

or

$$x - z = 1$$

$$-x - z = 0$$

To solve this system by Gauss-Jordan reduction, we would row-reduce the augmented matrix, which is A with the column $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ adjoined.

$$\left[\begin{array}{cc|c} 1 & -1 & 1 \\ -1 & -1 & 0 \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & x \\ 0 & 1 & z \end{array} \right]$$

To find the second column of A^{-1} we would similarly row-reduce the augmented matrix obtained by tacking on to A the second column of the identity matrix.

$$\left[\begin{array}{cc|c} 1 & -1 & 0 \\ -1 & -1 & 1 \end{array} \right] \rightarrow \left[\begin{array}{cc|c} 1 & 0 & y \\ 0 & 1 & w \end{array} \right]$$

The row operations used in doing these two reductions would be exactly the same. We could do both reductions simultaneously by “doubly augmenting” A , putting both columns of the identity matrix to the right of A .

$$\left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ -1 & -1 & 0 & 1 \end{array} \right] \rightarrow \left[\begin{array}{cc|cc} 1 & 0 & x & y \\ 0 & 1 & z & w \end{array} \right]$$

We carry out this reduction in the following example. ■

Example 2 Computing Matrix Inverse

Find the inverse of each matrix.

a. $P = \begin{bmatrix} 1 & -1 \\ -1 & -1 \end{bmatrix}$ **b.** $Q = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -2 & -1 \\ 3 & 0 & 0 \end{bmatrix}$

Solution

a. As described above, we put the matrix P on the left and the identity matrix I on the right to get a 2×4 matrix.

$$\left[\begin{array}{cc|cc} 1 & -1 & 1 & 0 \\ -1 & -1 & 0 & 1 \end{array} \right]$$

$\begin{matrix} P & & I \end{matrix}$

**using Technology**

See the Technology Guides at the end of the chapter to see how to invert a matrix using a TI-83/84 or Excel. You can also invert matrices with the Matrix Algebra Tool. Follow:

Chapter 3

→ Tools

→ Matrix Algebra Tool

To invert a matrix A you have entered, type A^{-1} or $A^{(-1)}$ in the formula box and press

“Compute.” To invert a matrix using the row-reduction method in Example 2, you can use either the browser-based Pivot and Gauss-Jordan Tool or the Excel-based Excel Pivot and Gauss-Jordan Tool, also accessible at

Chapter 3 → Tools

We now row reduce the whole matrix:

$$\begin{aligned} & \left[\begin{array}{cccc} 1 & -1 & 1 & 0 \\ -1 & -1 & 0 & 1 \end{array} \right] R_2 + R_1 \rightarrow \left[\begin{array}{cccc} 1 & -1 & 1 & 0 \\ 0 & -2 & 1 & 1 \end{array} \right] 2R_1 - R_2 \\ & \rightarrow \left[\begin{array}{cccc} 2 & 0 & 1 & -1 \\ 0 & -2 & 1 & 1 \end{array} \right] \begin{array}{l} \frac{1}{2}R_1 \\ -\frac{1}{2}R_2 \end{array} \rightarrow \left[\begin{array}{ccc|cc} 1 & 0 & 1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 1 & -\frac{1}{2} & -\frac{1}{2} \end{array} \right] \\ & \qquad \qquad \qquad I \qquad \qquad P^{-1} \end{aligned}$$

We have now solved the systems of linear equations that define the entries of P^{-1} . Thus,

$$P^{-1} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} \end{bmatrix}$$

b. The procedure to find the inverse of a 3×3 matrix (or larger) is just the same as for a 2×2 matrix. We place Q on the left and the identity matrix (now 3×3) on the right, and reduce.

$$\begin{aligned} & \begin{array}{ccc|ccc} Q & & & I & & \\ \hline 1 & 0 & 1 & 1 & 0 & 0 \\ 2 & -2 & -1 & 0 & 1 & 0 \\ 3 & 0 & 0 & 0 & 0 & 1 \end{array} R_2 - 2R_1 \rightarrow \begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & -2 & -3 & -2 & 1 & 0 \\ 0 & 0 & -3 & -3 & 0 & 1 \end{array} \begin{array}{l} 3R_1 + R_3 \\ R_2 - R_3 \end{array} \rightarrow \\ & \begin{array}{ccc|ccc} 3 & 0 & 0 & 0 & 0 & 1 \\ 0 & -2 & 0 & 1 & 1 & -1 \\ 0 & 0 & -3 & -3 & 0 & 1 \end{array} \begin{array}{l} \frac{1}{3}R_1 \\ -\frac{1}{2}R_2 \\ -\frac{1}{3}R_3 \end{array} \rightarrow \begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & \frac{1}{3} \\ 0 & 1 & 0 & -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 & 1 & 0 & -\frac{1}{3} \end{array} \\ & \qquad \qquad \qquad I \qquad \qquad Q^{-1} \end{aligned}$$

Thus,

$$Q^{-1} = \begin{bmatrix} 0 & 0 & \frac{1}{3} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ 1 & 0 & -\frac{1}{3} \end{bmatrix}$$

We have already checked that P^{-1} is the inverse of P . You should also check that Q^{-1} is the inverse of Q .

The method we used in Example 2 can be summarized as follows:

Inverting an $n \times n$ Matrix

In order to determine whether an $n \times n$ matrix A is invertible or not, and to find A^{-1} if it does exist, follow this procedure:

1. Write down the $n \times 2n$ matrix $[A|I]$ (this is A with the $n \times n$ identity matrix set next to it).
2. Row-reduce $[A|I]$.
3. If the reduced form is $[I|B]$ (i.e., has the identity matrix in the left part), then A is invertible and $B = A^{-1}$. If you cannot obtain I in the left part, then A is singular. (See Example 3 below.)

Although there is a general formula for the inverse of a matrix, it is not a simple one. In fact, using the formula for anything larger than a 3×3 matrix is so inefficient that the row-reduction procedure is the method of choice even for computers. However, the general formula is very simple for the special case of 2×2 matrices:

Formula for the Inverse of a 2×2 Matrix

The inverse of a 2×2 matrix is

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}, \quad \text{provided } ad - bc \neq 0$$

If the quantity $ad - bc$ is zero, then the matrix is singular (noninvertible). The quantity $ad - bc$ is called the **determinant** of the matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

quick Examples

- $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}^{-1} = \frac{1}{(1)(4) - (2)(3)} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = -\frac{1}{2} \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{bmatrix}$.
- $\begin{bmatrix} 1 & -1 \\ 2 & -2 \end{bmatrix}$ has determinant $ad - bc = (1)(-2) - (-1)(2) = 0$ and so is singular.

The formula for the inverse of a 2×2 matrix can be obtained using the technique of row reduction. (See the Communication and Reasoning Exercises at the end of the section.)

As we have mentioned above, not every square matrix has an inverse, as we see in the next example.

Example 3 Singular 3×3 Matrix

Find the inverse of the matrix $S = \begin{bmatrix} 1 & 1 & 2 \\ -2 & 0 & 4 \\ 3 & 1 & -2 \end{bmatrix}$, if it exists.

Solution We proceed as before.

$$\begin{array}{c} S \qquad I \\ \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ -2 & 0 & 4 & 0 & 1 & 0 \\ 3 & 1 & -2 & 0 & 0 & 1 \end{array} \right] \begin{array}{l} R_2 + 2R_1 \\ R_3 - 3R_1 \end{array} \rightarrow \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 8 & 2 & 1 & 0 \\ 0 & -2 & -8 & -3 & 0 & 1 \end{array} \right] \begin{array}{l} 2R_1 - R_2 \\ R_3 + R_2 \end{array} \\ \rightarrow \left[\begin{array}{ccc|ccc} 2 & 0 & -4 & 0 & -1 & 0 \\ 0 & 2 & 8 & 2 & 1 & 0 \\ 0 & 0 & 0 & -1 & 1 & 1 \end{array} \right] \end{array}$$

We stopped here, even though the reduction is incomplete, because there is *no hope* of getting the identity on the left-hand side. Completing the row reduction will not change the three zeros in the bottom row. So what is wrong? Nothing. As in Example 1, we have here a singular matrix. Any square matrix that, after row reduction, winds up with a row of zeros is singular (see Exercise 77).

† *Before we go on...* In practice, deciding whether a given matrix is invertible or singular is easy: simply try to find its inverse. If the process works, then the matrix is invertible, and we get its inverse. If the process fails, then the matrix is singular. If you try to invert a singular matrix using a spreadsheet, calculator, or computer program, you should get an error. Sometimes, instead of an error, you will get a spurious answer due to round-off errors in the device. ■

Using the Inverse to Solve a System of n Linear Equations in n Unknowns

Having used systems of equations and row reduction to find matrix inverses, we will now use matrix inverses to solve systems of equations. Recall that, at the end of the previous section, we saw that a system of linear equations could be written in the form

$$AX = B$$

where A is the coefficient matrix, X is the column matrix of unknowns, and B is the column matrix of right-hand sides. Now suppose that there are as many unknowns as equations, so that A is a square matrix, and suppose that A is invertible. The object is to solve for the matrix X of unknowns, so we multiply both sides of the equation by the inverse A^{-1} of A , getting

$$A^{-1}AX = A^{-1}B$$

Notice that we put A^{-1} on the left on both sides of the equation. Order matters when multiplying matrices, so we have to be careful to do the same thing to both sides of the equation. But now $A^{-1}A = I$, so we can rewrite the last equation as

$$IX = A^{-1}B$$

Also, $IX = X$ (I being the identity matrix), so we really have

$$X = A^{-1}B$$

and we have solved for X !

Moreover, we have shown that, if A is invertible and $AX = B$, then the only *possible* solution is $X = A^{-1}B$. We should check that $A^{-1}B$ is actually a solution by substituting back into the original equation.

$$AX = A(A^{-1}B) = (AA^{-1})B = IB = B$$

Thus, $X = A^{-1}B$ is a solution and is the only solution. Therefore, if A is invertible, $AX = B$ has exactly one solution.

On the other hand, if $AX = B$ has no solutions or has infinitely many solutions, we can conclude that A is not invertible (why?). To summarize:

Solving the Matrix Equation $AX = B$

If A is an invertible matrix, then the matrix equation $AX = B$ has the unique solution

$$X = A^{-1}B$$

quick Example

The system of linear equations

$$2x \quad + z = 9$$

$$2x + y - z = 6$$

$$3x + y - z = 9$$

can be written as $AX = B$, where

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & -1 \\ 3 & 1 & -1 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 9 \\ 6 \\ 9 \end{bmatrix}$$

The matrix A is invertible with inverse

$$A^{-1} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 5 & -4 \\ 1 & 2 & -2 \end{bmatrix} \quad \text{You should check this}$$

Thus,

$$X = A^{-1}B = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 5 & -4 \\ 1 & 2 & -2 \end{bmatrix} \begin{bmatrix} 9 \\ 6 \\ 9 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}$$

so that $(x, y, z) = (3, 3, 3)$ is the (unique) solution to the system.

Example 4 Solving Systems of Equations Using an Inverse

Solve the following three systems of equations.

$$\begin{array}{lll} \mathbf{a.} & 2x & +z = 1 \\ & 2x + y - z = 1 \\ & 3x + y - z = 1 \\ \mathbf{b.} & 2x & +z = 0 \\ & 2x + y - z = 1 \\ & 3x + y - z = 2 \\ \mathbf{c.} & 2x & +z = 0 \\ & 2x + y - z = 0 \\ & 3x + y - z = 0 \end{array}$$

Solution We *could* go ahead and row-reduce all three augmented matrices as we did in Chapter 2, but this would require a lot of work. Notice that the coefficients are the same in all three systems. In other words, we can write the three systems in matrix form as

$$\mathbf{a.} \quad AX = B \quad \mathbf{b.} \quad AX = C \quad \mathbf{c.} \quad AX = D$$

where the matrix A is the same in all three cases:

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 2 & 1 & -1 \\ 3 & 1 & -1 \end{bmatrix}$$

Now the solutions to these systems are

$$\mathbf{a.} \quad X = A^{-1}B \quad \mathbf{b.} \quad X = A^{-1}C \quad \mathbf{c.} \quad X = A^{-1}D$$

so the main work is the calculation of the single matrix A^{-1} , which we have already noted (Quick Example above) is

$$A^{-1} = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 5 & -4 \\ 1 & 2 & -2 \end{bmatrix}$$

Thus, the three solutions are

$$\begin{array}{l} \mathbf{a.} \quad X = A^{-1}B = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 5 & -4 \\ 1 & 2 & -2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \\ \mathbf{b.} \quad X = A^{-1}C = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 5 & -4 \\ 1 & 2 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ -2 \end{bmatrix} \\ \mathbf{c.} \quad X = A^{-1}D = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 5 & -4 \\ 1 & 2 & -2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \end{array}$$



using Technology

See the Technology Guides at the end of the chapter to see how to do these calculations using a TI-83/84 or Excel. If, instead, you use the online Matrix Algebra Tool, you can use the formulas $A^{-1}B$, $A^{-1}C$, and $A^{-1}D$ to compute these products after entering the matrices A , B and C as usual (See p. 175.)

† *Before we go on...* We have been speaking of *the* inverse of a matrix A . Is there only one? It is not hard to prove that a matrix A cannot have more than one inverse: If B and C were both inverses of A , then

$$\begin{aligned} B &= BI && \text{Property of the identity} \\ &= B(AC) && \text{Since } C \text{ is an inverse of } A \\ &= (BA)C && \text{Associative law} \\ &= IC && \text{Since } B \text{ is an inverse of } A \\ &= C && \text{Property of the identity} \end{aligned}$$

In other words, if B and C were both inverses of A , then B and C would have to be equal. ■

FAQs Which Method to Use in Solving a System

Q: Now we have two methods to solve a system of linear equations $AX = B$: (1) Compute $X = A^{-1}B$, or (2) row-reduce the augmented matrix. Which is the better method?

A: Each method has its advantages and disadvantages. Method (1), as we have seen, is very efficient when you must solve several systems of equations with the same coefficients, but it works only when the coefficient matrix is *square* (meaning that you have the same number of equations as unknowns) and *invertible* (meaning that there is a unique solution). The row-reduction method will work for all systems. Moreover, for all but the smallest systems, the most efficient way to find A^{-1} is to use row reduction. Thus, in practice, the two methods are essentially the same when both apply. ■

3.3 EXERCISES

● denotes basic skills exercises

◆ denotes challenging exercises

tech Ex indicates exercises that should be solved using technology

In Exercises 1–6, determine whether or not the given pairs of matrices are inverse pairs. *hint* [see Quick Examples p. 199]

1. ● $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

2. ● $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$, $B = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{3} \end{bmatrix}$

3. ● $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

4. ● $A = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}$

5. ● $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} a^{-1} & 0 & 0 \\ 0 & b^{-1} & 0 \\ 0 & 0 & 0 \end{bmatrix}$ ($a, b \neq 0$)

6. ● $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$, $B = \begin{bmatrix} a^{-1} & 0 & 0 \\ 0 & b^{-1} & 0 \\ 0 & 0 & c^{-1} \end{bmatrix}$ ($a, b, c \neq 0$)

In Exercises 7–26, use row reduction to find the inverses of the given matrices if they exist, and check your answers by multiplication. *hint* [see Example 2]

7. ● $\begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}$ 8. ● $\begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ 9. ● $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ 10. ● $\begin{bmatrix} 4 & 0 \\ 0 & 2 \end{bmatrix}$

11. ● $\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ 12. ● $\begin{bmatrix} 3 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$ 13. ● $\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$ 14. ● $\begin{bmatrix} 1 & 1 \\ 6 & 6 \end{bmatrix}$

15. ● $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ 16. ● $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$ 17. ● $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix}$

18. ● $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 2 & 3 \\ 1 & 0 & 1 \end{bmatrix}$ 19. ● $\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \\ 1 & 2 & 3 \end{bmatrix}$ 20. ● $\begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix}$

● basic skills ◆ challenging **tech** Ex technology exercise

21. ●
$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

22. ●
$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

23. ●
$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ -1 & 1 & 0 & 1 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

24. ●
$$\begin{bmatrix} 0 & 1 & 1 & 0 \\ -1 & 1 & 1 & 1 \\ -1 & 1 & 0 & 1 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

25. ●
$$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

26. ●
$$\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

In Exercises 27–34, compute the determinant of the given matrix. If the determinant is nonzero, use the formula for inverting a 2×2 matrix to calculate the inverse of the given matrix.

hint [see Quick Examples p. 202]

27. ●
$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

28. ●
$$\begin{bmatrix} 4 & 1 \\ 0 & 2 \end{bmatrix}$$

29. ●
$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

30. ●
$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

31. ●
$$\begin{bmatrix} \frac{1}{6} & -\frac{1}{6} \\ 0 & \frac{1}{6} \end{bmatrix}$$

32. ●
$$\begin{bmatrix} 2 & 1 \\ 4 & 2 \end{bmatrix}$$

33. ●
$$\begin{bmatrix} 1 & 0 \\ \frac{3}{4} & 0 \end{bmatrix}$$

34. ●
$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

tech Ex In Exercises 35–42, use technology to find the inverse of the given matrix (when it exists). Round all entries in your answer to two decimal places. [Caution: Because of rounding errors, technology sometimes produces an “inverse” of a singular matrix. These can be often recognized by their huge entries.]

35. ●
$$\begin{bmatrix} 1.1 & 1.2 \\ 1.3 & -1 \end{bmatrix}$$

36. ●
$$\begin{bmatrix} 0.1 & -3.2 \\ 0.1 & -1.5 \end{bmatrix}$$

37. ●
$$\begin{bmatrix} 3.56 & 1.23 \\ -1.01 & 0 \end{bmatrix}$$

38. ●
$$\begin{bmatrix} 9.09 & -5.01 \\ 1.01 & 2.20 \end{bmatrix}$$

39. ●
$$\begin{bmatrix} 1.1 & 3.1 & 2.4 \\ 1.7 & 2.4 & 2.3 \\ 0.6 & -0.7 & -0.1 \end{bmatrix}$$

40. ●
$$\begin{bmatrix} 2.1 & 2.4 & 3.5 \\ 6.1 & -0.1 & 2.3 \\ -0.3 & -1.2 & 0.1 \end{bmatrix}$$

41. ●
$$\begin{bmatrix} 0.01 & 0.32 & 0 & 0.04 \\ -0.01 & 0 & 0 & 0.34 \\ 0 & 0.32 & -0.23 & 0.23 \\ 0 & 0.41 & 0 & 0.01 \end{bmatrix}$$

42. ●
$$\begin{bmatrix} 0.01 & 0.32 & 0 & 0.04 \\ -0.01 & 0 & 0 & 0.34 \\ 0 & 0.32 & -0.23 & 0.23 \\ 0.01 & 0.96 & -0.23 & 0.65 \end{bmatrix}$$

In Exercises 43–48, use matrix inversion to solve the given systems of linear equations. (You previously solved all of these systems using row reduction in Chapter 2.) *hint* [see Quick Example p. 203]

43. ●
$$\begin{cases} x + y = 4 \\ x - y = 1 \end{cases}$$

44. ●
$$\begin{cases} 2x + y = 2 \\ 2x - 3y = 2 \end{cases}$$

45. ●
$$\begin{cases} \frac{x}{3} + \frac{y}{2} = 0 \\ \frac{x}{2} + y = -1 \end{cases}$$

46. ●
$$\begin{cases} \frac{2x}{3} - \frac{y}{2} = \frac{1}{6} \\ \frac{x}{2} - \frac{y}{2} = -1 \end{cases}$$

47. ●
$$\begin{cases} -x + 2y - z = 0 \\ -x - y + 2z = 0 \\ 2x - z = 6 \end{cases}$$

48. ●
$$\begin{cases} x + 2y = 4 \\ y - z = 0 \\ x + 3y - 2z = 5 \end{cases}$$

In Exercises 49 and 50, use matrix inversion to solve each collection of systems of linear equations. *hint* [see Example 4]

49. ● a.
$$\begin{cases} -x - 4y + 2z = 4 \\ x + 2y - z = 3 \\ x + y - z = 8 \end{cases}$$

b.
$$\begin{cases} -x - 4y + 2z = 0 \\ x + 2y - z = 3 \\ x + y - z = 2 \end{cases}$$

c.
$$\begin{cases} -x - 4y + 2z = 0 \\ x + 2y - z = 0 \\ x + y - z = 0 \end{cases}$$

50. ● a.
$$\begin{cases} -x - 4y + 2z = 8 \\ x - z = 3 \\ x + y - z = 8 \end{cases}$$

b.
$$\begin{cases} -x - 4y + 2z = 8 \\ x - z = 3 \\ x + y - z = 2 \end{cases}$$

c.
$$\begin{cases} -x - 4y + 2z = 0 \\ x - z = 0 \\ x + y - z = 0 \end{cases}$$

Applications

Some of the following exercises are similar or identical to exercises and examples in Chapter 2. Use matrix inverses to find the solutions. We suggest you invert some of the matrices by hand, and others using technology.

51. ● **Nutrition** A four-ounce serving of Campbell's® Pork & Beans contains 5 grams of protein and 21 grams of carbohydrates.¹⁶ A typical slice of “lite” rye bread contains 4 grams of protein and 12 grams of carbohydrates.

- I am planning a meal of “beans-on-toast” and I want it to supply 20 grams of protein and 80 grams of carbohydrates. How should I prepare my meal?
- If I require A grams of protein and B grams of carbohydrates, give a formula that tells me how many slices of bread and how many servings of Pork & Beans to use.

52. ● **Nutrition** According to the nutritional information on a package of Honey Nut Cheerios® brand cereal, each 1-ounce serving of Cheerios contains 3 grams protein and 24 grams carbohydrates.¹⁷ Each half-cup serving of enriched skim milk contains 4 grams protein and 6 grams carbohydrates.

- I am planning a meal of cereal and milk and I want it to supply 26 grams of protein, and 78 grams of carbohydrates. How should I prepare my meal?

¹⁶ According to the label information on a 16-oz can.

¹⁷ Actually, it is 23 grams carbohydrates. We made it 24 grams to simplify the calculation.

b. If I require A grams of protein and B grams of carbohydrates, give a formula that tells me how many servings of milk and Cheerios to use.

53. ● **Resource Allocation** You manage an ice cream factory that makes three flavors: Creamy Vanilla, Continental Mocha, and Succulent Strawberry. Into each batch of Creamy Vanilla go two eggs, one cup of milk, and two cups of cream. Into each batch of Continental Mocha go one egg, one cup of milk, and two cups of cream. Into each batch of Succulent Strawberry go one egg, two cups of milk, and one cup of cream. Your stocks of eggs, milk, and cream vary from day to day. How many batches of each flavor should you make in order to use up all of your ingredients if you have the following amounts in stock?

- 350 eggs, 350 cups of milk, and 400 cups of cream
- 400 eggs, 500 cups of milk, and 400 cups of cream
- A eggs, B cups of milk, and C cups of cream

54. ● **Resource Allocation** The Arctic Juice Company makes three juice blends: PineOrange, using 2 quarts of pineapple juice and 2 quarts of orange juice per gallon; PineKiwi, using 3 quarts of pineapple juice and 1 quart of kiwi juice per gallon; and OrangeKiwi, using 3 quarts of orange juice and 1 quart of kiwi juice per gallon. The amount of each kind of juice the company has on hand varies from day to day. How many gallons of each blend can it make on a day with the following stocks?

- 800 quarts of pineapple juice, 650 quarts of orange juice, 350 quarts of kiwi juice.
- 650 quarts of pineapple juice, 800 quarts of orange juice, 350 quarts of kiwi juice.
- A quarts of pineapple juice, B quarts of orange juice, C quarts of kiwi juice.

Investing in Municipal Bond Funds Exercises 55 and 56 are based on the following data on three tax-exempt municipal bond funds.¹⁸

	2003 Yield
PNF (Pimco NY)	6%
FDMMX (Fidelity Spartan Mass)	5%
FFLIX (Fidelity Spartan Florida)	7%

55. ● You invested a total of \$9000 in the three funds at the beginning of 2003, including an equal amount in FDMMX and FFLIX. Your 2003 yield for the year from the first two funds amounted to \$400. How much did you invest in each of the three funds?

56. ● You invested a total of \$6000 in the three funds at the beginning of 2003, including an equal amount in PNF and FDMMX. Your total yields for 2003 amounted to \$360. How much did you invest in each of the three funds?

¹⁸ Yields are rounded. SOURCES: www.pimcofunds.com, www.fidelity.com. January 18, 2004.

Investing in Stocks Exercises 57 and 58 are based on the following data on three computer stocks.¹⁹

	Price	Dividend Yield
APPL (Apple Computer)	\$25	0.6%
HPQ (Hewlett Packard)	25	1.2
DELL	35	0

57. You invested a total of \$5800 in Apple, Hewlett Packard, and Dell shares at the above prices, and expected to earn \$21 in annual dividends. If you purchased a total of 200 shares, how many shares of each stock did you purchase?

58. You invested a total of \$6750 in Apple, Hewlett Packard, and Dell shares at the above prices, and expected to earn \$45 in annual dividends. If you purchased a total of 250 shares, how many shares of each stock did you purchase?

59. ♦ **tech**Ex **Population Movement** In 2003, the population of the U.S., broken down by regions, was 53.3 million in the Northeast, 64.0 million in the Midwest, 101.6 million in the South, and 65.4 million in the West.²⁰ The matrix P below shows the population movement during the period 2003–2004. (Thus, 98.79% of the population in the Northeast stayed there, while 0.20% of the population in the Northeast moved to the Midwest, and so on.)

$$P = \begin{matrix} & \begin{matrix} \text{To} & \text{To} & \text{To} & \text{To} \\ \text{NE} & \text{MW} & \text{S} & \text{W} \end{matrix} \\ \begin{matrix} \text{From NE} \\ \text{From MW} \\ \text{From S} \\ \text{From W} \end{matrix} & \begin{bmatrix} 0.9879 & 0.0020 & 0.0081 & 0.0019 \\ 0.0014 & 0.9895 & 0.0063 & 0.0028 \\ 0.0027 & 0.0025 & 0.9927 & 0.0022 \\ 0.0010 & 0.0030 & 0.0050 & 0.9909 \end{bmatrix} \end{matrix}$$

Set up the 2003 population figures as a row vector. Assuming that these percentages also describe the population movements from 2002 to 2003, show how matrix inversion and multiplication allow you to compute the population in each region in 2002. (Round all answers to the nearest 0.1 million.)

60. ♦ **tech**Ex **Population Movement** Assume that the percentages given in the preceding exercise also describe the population movements from 2001 to 2002. Show how matrix inversion and multiplication allow you to compute the population in each region in 2001. (Round all answers to the nearest 0.1 million.)

61. ♦ **tech**Ex **Rotations** If a point (x, y) in the plane is rotated counterclockwise through an angle of 45° , its new coordinates

¹⁹ Stocks were trading at or near the given prices in January, 2004. Dividends are rounded, based on values in January 18, 2004. SOURCE: <http://money.excite.com>, January 18, 2004.

²⁰ Note that this exercise ignores migration into or out of the country. The internal migration figures and 2004 population figures are accurate. SOURCE: U.S. Census Bureau, Current Population Survey, 2004 Annual Social and Economic Supplement, Internet release date June 23, 2005. www.census.gov.

(x', y') are given by

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = R \begin{bmatrix} x \\ y \end{bmatrix}$$

where R is the 2×2 matrix $\begin{bmatrix} a & -a \\ a & a \end{bmatrix}$ and $a = \sqrt{1/2} \approx 0.7071$.

- If the point $(2, 3)$ is rotated counterclockwise through an angle of 45° , what are its (approximate) new coordinates?
 - Multiplication by what matrix would result in a counterclockwise rotation of 90° ? 135° ? (Express the matrices in terms of R . Hint: Think of a rotation through 90° as two successive rotations through 45° .)
 - Multiplication by what matrix would result in a *clockwise* rotation of 45° ?
62. ♦ **tech**Ex **Rotations** If a point (x, y) in the plane is rotated counterclockwise through an angle of 60° , its new coordinates are given by

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = S \begin{bmatrix} x \\ y \end{bmatrix}$$

where S is the 2×2 matrix $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ and $a = 1/2$ and $b = \sqrt{3}/4 \approx 0.8660$.

- If the point $(2, 3)$ is rotated counterclockwise through an angle of 60° , what are its (approximate) new coordinates?
- Referring to Exercise 61, multiplication by what matrix would result in a counterclockwise rotation of 105° ? (Express the matrices in terms of S and the matrix R from Exercise 61. Hint: Think of a rotation through 105° as a rotation through 60° followed by one through 45° .)
- Multiplication by what matrix would result in a *clockwise* rotation of 60° ?

techEx **Encryption** Matrices are commonly used to encrypt data. Here is a simple form such an encryption can take. First, we represent each letter in the alphabet by a number. For example, if we take <space> = 0, $A = 1$, $B = 2$, and so on,

“ABORT MISSION” becomes

$$[1 \ 2 \ 15 \ 18 \ 20 \ 0 \ 13 \ 9 \ 19 \ 19 \ 9 \ 15 \ 14]$$

To encrypt this coded phrase, we use an invertible matrix of any size with integer entries. For instance, let us take A to be the 2×2 matrix $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$. We can first arrange the coded sequence of numbers in the form of a matrix with two rows (using zero in the last place if we have an odd number of characters) and then multiply on the left by A .

$$\begin{aligned} \text{Encrypted Matrix} &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 15 & 20 & 13 & 19 & 9 & 14 \\ 2 & 18 & 0 & 9 & 19 & 15 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 5 & 51 & 20 & 31 & 57 & 39 & 14 \\ 11 & 117 & 60 & 75 & 133 & 87 & 42 \end{bmatrix} \end{aligned}$$

which we can also write as

$$[5 \ 11 \ 51 \ 117 \ 20 \ 60 \ 31 \ 75 \ 57 \ 133 \ 39 \ 87 \ 14 \ 42].$$

To decipher the encoded message, multiply the encrypted matrix by A^{-1} .

63. Use the matrix A to encode the phrase “GO TO PLAN B.”

64. Use the matrix A to encode the phrase “ABANDON SHIP.”
65. Decode the following message, which was encrypted using the matrix A .

$$[33 \ 69 \ 54 \ 126 \ 11 \ 27 \ 20 \ 60 \ 29 \ 59 \ 65 \ 149 \ 41 \ 87]$$

66. Decode the following message, which was encrypted using the matrix A .

$$[59 \ 141 \ 43 \ 101 \ 7 \ 21 \ 29 \ 59 \ 65 \ 149 \ 41 \ 87]$$

Communication and Reasoning Exercises

67. ● Multiple Choice: If A and B are square matrices with $AB = I$ and $BA = I$, then
- B is the inverse of A .
 - A and B must be equal.
 - A and B must both be singular.
 - At least one of A and B is singular.
68. ● Multiple Choice: If A is a square matrix with $A^3 = I$, then
- A must be the identity matrix.
 - A is invertible.
 - A is singular.
 - A is both invertible and singular.
69. ● What can you say about the inverse of a 2×2 matrix of the form $\begin{bmatrix} a & b \\ a & b \end{bmatrix}$?
70. ● If you think of numbers as 1×1 matrices, which numbers are invertible 1×1 matrices?
71. Use matrix multiplication to check that the inverse of a general 2×2 matrix is given by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad (\text{provided } ad - bc \neq 0)$$

72. ♦ Derive the formula in Exercise 71 using row reduction. (Assume that $ad - bc \neq 0$.)

73. A **diagonal** matrix D has the following form.

$$D = \begin{bmatrix} d_1 & 0 & 0 & \dots & 0 \\ 0 & d_2 & 0 & \dots & 0 \\ 0 & 0 & d_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & d_n \end{bmatrix}$$

When is D singular? Why?

74. If a square matrix A row-reduces to the identity matrix, must it be invertible? If so, say why, and if not, give an example of such a (singular) matrix.
75. If A and B are invertible, check that $B^{-1}A^{-1}$ is the inverse of AB .
76. Solve the matrix equation $A(B + CX) = D$ for X . (You may assume that all the matrices are square and invertible.)
77. ♦ In Example 3 we said that, if a square matrix A row-reduces to a matrix with a row of zeros is singular. Why?
78. ♦ Your friend has two square matrices A and B , neither of them the zero matrix, with the property that AB is the zero matrix. You immediately tell him that neither A nor B can possibly be invertible. How can you be so sure?

3.4 Game Theory

It frequently happens that you are faced with making a decision or choosing a best strategy from several possible choices. For instance, you might need to decide whether to invest in stocks or bonds, whether to cut prices of the product you sell, or what offensive play to use in a football game. In these examples, the result depends on something you cannot control. In the first case, your success depends on the future behavior of the economy. In the second case, it depends in part on whether your competitors also cut prices, and in the third case, it depends on the defensive strategy chosen by the opposing team.

We can model situations like these using **game theory**. We represent the various options and payoffs in a matrix and can then calculate the best single strategy or combination of strategies using matrix algebra and other techniques.

Game theory is very new compared with most of the mathematics you learn. It was invented in the 1920s by the noted mathematicians Emile Borel (1871–1956) and John von Neumann (1903–1957). Game theory's connection with linear programming was discovered even more recently, in 1947, by von Neumann, and further advances were made by the mathematician John Nash²¹ (1928–), for which he received the 1994 Nobel Prize for Economics.

The Payoff Matrix and Expected Value of a Game

We have probably all played the simple game “Paper, Scissors, Rock” at some time in our lives. It goes as follows: There are two players—let us call them A and B—and at each turn, both players produce, by a gesture of the hand, either paper, a pair of scissors, or a rock. Rock beats scissors (since a rock can crush scissors), but is beaten by paper (since a rock can be covered by paper), while scissors beat paper (since scissors can cut paper). The round is a draw if both A and B show the same item. We could turn this into a betting game if, at each turn, we require the loser to pay the winner 1¢. For instance, if A shows a rock and B shows paper, then A pays B 1¢.

Paper, Scissors, Rock is an example of a **two-person zero-sum game**. It is called a zero-sum game because each player's loss is equal to the other player's gain.²² We can represent this game by a matrix, called the **payoff matrix**.

$$\begin{array}{c} \mathbf{B} \\ \begin{array}{c} p \quad s \quad r \\ \mathbf{A} \begin{array}{l} p \left[\begin{array}{ccc} 0 & -1 & 1 \\ s \left[\begin{array}{ccc} 1 & 0 & -1 \\ r \left[\begin{array}{ccc} -1 & 1 & 0 \end{array} \right] \end{array} \right] \end{array} \right] \end{array} \end{array}$$

or just

$$P = \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

if we choose to omit the labels. In the payoff matrix, Player A's options, or **moves**, are listed on the left, while Player B's options are listed on top. We think of A as playing the rows and B as playing the columns. Positive numbers indicate a win for the row player,

²¹ Nash's turbulent life is the subject of the biography “A Beautiful Mind” by Sylvia Nasar (Simon & Schuster, 1998). The 2001 Academy Award-winning movie of the same title is a somewhat fictionalized account.

²² An example of a *nonzero sum game* would be one in which the government taxed the earnings of the winner. In that case the winner's gain would be less than the loser's loss.

while negative numbers indicate a loss for the row player. Thus, for example, the p, s entry represents the outcome if A plays p (paper) and B plays s (scissors). In this event, B wins, and the -1 entry there indicates that A loses 1¢. (If that entry were -2 instead, it would have meant that A loses 2¢.)

Two-Person Zero-Sum Game

A **two-person zero-sum game** is one in which one player's loss equals the other's gain. We assume that the outcome is determined by each player's choice from among a fixed, finite set of moves. If Player A has m moves to choose from and Player B has n , we can represent the game using the **payoff matrix**, the $m \times n$ matrix showing the result of each possible pair of choices of moves.

In each round of the game, the way a player chooses a move is called a **strategy**. A player using a **pure strategy** makes the same move each round of the game. For example, if a player in the above game chooses to play scissors at each turn, then that player is using the pure strategy s . A player using a **mixed strategy** chooses each move a certain percentage of the time in a random fashion; for instance, Player A might choose to play p 50% of the time, and each of s and r 25% of the time.

Our ultimate goal is to be able to determine which strategy is best for each player to use. To do that, we need to know how to evaluate strategies. The fundamental calculation we need is that of the **expected value** of a pair of strategies. Let's look at a simple example.

Example 1 Expected Value

Consider the following game:

$$\begin{array}{c} \mathbf{B} \\ a \quad b \\ \mathbf{A} \begin{array}{l} p \\ q \end{array} \begin{bmatrix} 3 & -1 \\ -2 & 3 \end{bmatrix} \end{array}$$

Player A decides to pick moves at random, choosing to play p 75% of the time and q 25% of the time. Player B also picks moves at random, choosing a 20% of the time and b 80% of the time. On average, how much does A expect to win or lose?

Solution Suppose they play the game 100 times. Each time they play there are four possible outcomes:

Case 1: *A plays p , B plays a .*

Because A plays p only 75% of the time and B plays a only 20% of the time, we expect this case to occur $0.75 \times 0.20 = 0.15$, or 15% of the time, or 15 times out of 100. Each time this happens, A gains 3 points, so we get a contribution of $15 \times 3 = 45$ points to A's total winnings.

Case 2: *A plays p , B plays b .*

Because A plays p only 75% of the time and B plays b only 80% of the time, we expect this case to occur $0.75 \times 0.80 = 0.60$, or 60 times out of 100. Each time this happens, A loses 1 point, so we get a contribution of: $60 \times -1 = -60$ to A's total winnings.

Case 3: *A plays q , B plays a .*

This case occurs $0.25 \times 0.20 = 0.05$, or 5 out of 100 times, with a loss of 2 points to A each time, giving a contribution of $5 \times -2 = -10$ to A's total winnings.

Case 4: *A plays q, B plays b.*

This case occurs $0.25 \times 0.80 = 0.20$, or 20 out of 100 times, with a gain of 3 points to A each time, giving a contribution of $20 \times 3 = 60$ to A's total winnings.

Summing to get A's total winnings and then dividing by the number of times the game is played gives the average value of

$$(45 - 60 - 10 + 60)/100 = 0.35$$

so that A can expect to win an average of 0.35 points per play of the game. We call 0.35 the **expected value** of the game when A and B use these particular strategies.

This calculation was somewhat tedious and it would only get worse if A and B had many moves to choose from. There is a far more convenient way of doing exactly the same calculation, using matrix multiplication: We start by representing the player's strategies as matrices. For reasons to become clear in a moment, we record A's strategy as a row matrix:

$$R = [0.75 \quad 0.25]$$

We record B's strategy as a column matrix:

$$C = \begin{bmatrix} 0.20 \\ 0.80 \end{bmatrix}$$

(We will sometimes write column vectors using transpose notation, writing, for example, $[0.20 \quad 0.80]^T$ for the column above, to save space.) Now: *the expected value is the matrix product RPC where P is the payoff matrix!*

$$\begin{aligned} \text{Expected value} = RPC &= [0.75 \quad 0.25] \begin{bmatrix} 3 & -1 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} 0.20 \\ 0.80 \end{bmatrix} \\ &= [1.75 \quad 0] \begin{bmatrix} 0.20 \\ 0.80 \end{bmatrix} = [0.35] \end{aligned}$$

Why does this work? Write out the arithmetic involved in the matrix product RPC to see what we calculated:

$$\begin{aligned} &[0.75 \times 3 + 0.25 \times (-2)] \times 0.20 + [0.75 \times (-1) + 0.25 \times 3] \times 0.80 \\ &= 0.75 \times 3 \times 0.20 + 0.25 \times (-2) \times 0.20 + 0.75 \times (-1) \times 0.80 + 0.25 \times 3 \times 0.80 \\ &= \text{Case 1} \quad + \quad \text{Case 3} \quad + \quad \text{Case 2} \quad + \quad \text{Case 4} \end{aligned}$$

So, the matrix product does all at once the various cases we considered above.

To summarize what we just saw:

The Expected Value of a Game for Mixed Strategies R and C

The **expected value of a game for given mixed strategies** is the average payoff that occurs if the game is played a large number of times with the row and column players using the given strategies.

To compute the expected value of a game:

1. Write the row player's mixed strategy as a row matrix R .
2. Write the column player's mixed strategy as a column matrix C .
3. Calculate the product RPC , where P is the payoff matrix. This product is a 1×1 matrix whose entry is the expected value e .

**using Technology**

The use of technology becomes indispensable when we need to do several calculations or when the matrices involved are big. See the end-of-chapter Technology Guide notes on Example 3 in Section 3.2 for instructions on multiplying matrices using a TI-83/84 or Excel. Online, use the Matrix Algebra Tool.

quick Example

Consider “Paper, Scissors, Rock,”

$$\mathbf{A} = \begin{matrix} & \mathbf{B} \\ & \begin{matrix} p & s & r \end{matrix} \\ \begin{matrix} p \\ s \\ r \end{matrix} & \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \end{matrix}$$

Suppose that the row player plays *paper* half the time and each of the other two strategies a quarter of the time, and the column player always plays *scissors*. We write

$$R = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}, \quad C = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

so,

$$e = RPC = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = -\frac{1}{4}$$

Thus, player A loses an average of once every four plays.

Solving a Game

Now that we know how to evaluate particular strategies, we want to find the *best* strategy. The next example takes us another step toward that goal.

Example 2 Television Ratings Wars

Commercial TV station RTV and cultural station CTV are competing for viewers in the Tuesday prime-time 9–10 PM time slot. RTV is trying to decide whether to show a sitcom, a docudrama, a reality show, or a movie, while CTV is thinking about either a nature documentary, a symphony concert, a ballet, or an opera. A television rating company estimates the payoffs for the various alternatives as follows. (Each point indicates a shift of 1000 viewers from one channel to the other; thus, -2 indicates a shift of 2000 viewers from RTV to CTV.)

		CTV			
		Nature Doc.	Symphony	Ballet	Opera
RTV	Sitcom	2	1	-2	2
	Docudrama	-1	1	-1	2
	Reality Show	-2	0	0	1
	Movie	3	1	-1	1

- If RTV notices that CTV is showing nature documentaries half the time and symphonies the other half, what would RTV's best strategy be, and how many viewers would it gain if it followed this strategy?
- If, on the other hand, CTV notices that RTV is showing docudramas half the time and reality shows the other half, what would CTV's best strategy be, and how many viewers would it gain or lose if it followed this strategy?

Solution

- a.** We are given the matrix of the game, P , in the table above, and we are given CTV's strategy $C = [0.50 \ 0.50 \ 0 \ 0]^T$. We are not given RTV's strategy R . To say that RTV is looking for its best strategy is to say that it wants the expected value $e = RPC$ to be as large as possible. So, we take $R = [x \ y \ z \ t]$ and look for values for x , y , z , and t that make RPC as large as possible. First, we calculate e in terms of these unknowns:

$$\begin{aligned} e = RPC &= [x \ y \ z \ t] \begin{bmatrix} 2 & 1 & -2 & 2 \\ -1 & 1 & -1 & 2 \\ -2 & 0 & 0 & 1 \\ 3 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 0.50 \\ 0.50 \\ 0 \\ 0 \end{bmatrix} \\ &= [x \ y \ z \ t] \begin{bmatrix} 1.5 \\ 0 \\ -1 \\ 2 \end{bmatrix} = 1.5x - z + 2t \end{aligned}$$

Now, the unknowns x , y , z , and t must be nonnegative and add up to 1 (why?). Because t has the largest coefficient, 2, we'll get the best result by making it as large as possible, namely, $t = 1$, leaving $x = y = z = 0$. Thus, RTV's best strategy is $R = [0 \ 0 \ 0 \ 1]$. In other words, RTV should use the pure strategy of showing a movie every Tuesday evening. If it does so, the expected value will be

$$e = 1.5(0) - 0 + 2(1) = 2$$

so RTV can expect to gain 2000 viewers.

- b.** Here, we are given $R = [0 \ 0.50 \ 0.50 \ 0]$ and are not given CTV's strategy C , so this time we take $C = [x \ y \ z \ t]^T$ and calculate e :

$$\begin{aligned} e = RPC &= [0 \ 0.50 \ 0.50 \ 0] \begin{bmatrix} 2 & 1 & -2 & 2 \\ -1 & 1 & -1 & 2 \\ -2 & 0 & 0 & 1 \\ 3 & 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} \\ &= [-1.5 \ 0.5 \ -0.5 \ 1.5] \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = -1.5x + 0.5y - 0.5z + 1.5t \end{aligned}$$

Now, CTV wants e to be as *small* as possible (why?). Because x has the largest negative coefficient, CTV would like it to be as large as possible: $x = 1$, so the rest of the unknowns must be zero. Thus, CTV's best strategy is $C = [1 \ 0 \ 0 \ 0]^T$, that is, show a nature documentary every night. If it does so, the expected value will be

$$e = -1.5(1) + 0.5(0) - 0.5(0) + 1.5(0) = -1.5$$

so CTV can expect to gain 1500 viewers.

This example illustrates the fact that, no matter what mixed strategy one player selects, the other player can choose an appropriate *pure* counterstrategy in order to maximize its gain. How does this affect what decisions you should make as one of the players? If you were on the board of directors of RTV, you might reason as follows: Since for every mixed strategy you try, CTV can find a best counterstrategy (as in part (b)) it is in your company's best interest to select a mixed strategy that *minimizes* the effect of CTV's best counterstrategy. This is called the **minimax criterion**.

Minimax Criterion

A player using the **minimax criterion** chooses a strategy that, among all possible strategies, minimizes the effect of the other player's best counterstrategy. That is, an optimal (best) strategy according to the minimax criterion is one that minimizes the maximum damage the opponent can cause.

This criterion assumes that your opponent is determined to win. More precisely, it assumes the following.

Fundamental Principle of Game Theory

Each player tries to use its best possible strategy, and assumes that the other player is doing the same.

This principle is not always followed by every player. For example, one of the players may be nature and may choose its move at random, with no particular purpose in mind. In such a case, criteria other than the minimax criterion may be more appropriate. For example, there is the “maximax” criterion, which maximizes the maximum possible payoff (also known as the “reckless” strategy), or the criterion that seeks to minimize “regret” (the difference between the payoff you get and the payoff you *would have gotten* if you had known beforehand what was going to happen).²³ But, we shall assume here the fundamental principle and try to find optimal strategies under the minimax criterion.

Finding the optimal strategy is called **solving the game**. In general, solving a game can be done using linear programming, as we shall see in the next chapter. However, we can solve 2×2 games “by hand” as we shall see in the next example. First, we notice that some large games can be reduced to smaller games.

Consider the game in the preceding example, which had the following matrix:

$$P = \begin{bmatrix} 2 & 1 & -2 & 2 \\ -1 & 1 & -1 & 2 \\ -2 & 0 & 0 & 1 \\ 3 & 1 & -1 & 1 \end{bmatrix}$$

Compare the second and third columns through the eyes of the column player, CTV. Every payoff in the third column is as good as or better, from CTV's point of view, than the corresponding entry in the second column. Thus, no matter what RTV does, CTV will do better showing a ballet (third column) than a symphony (second column). We say that the third column **dominates** the second column. As far as CTV is concerned, we might as well forget about symphonies entirely, so we remove the second column. Similarly, the third column dominates the fourth, so we can remove the fourth column, too. This gives us a smaller game to work with:

$$P = \begin{bmatrix} 2 & -2 \\ -1 & -1 \\ -2 & 0 \\ 3 & -1 \end{bmatrix}$$

²³ See *Location in Space: Theoretical Perspectives in Economic Geography*, 3rd Edition, by Peter Dicken and Peter E. Lloyd, HarperCollins Publishers, 1990, pp. 276 ff.

Now compare the first and last rows. Every payoff in the last row is larger than the corresponding payoff in the first row, so the last row is always better to RTV. Again, we say that the last row dominates the first row, and we can discard the first row. Similarly, the last row dominates the second row, so we discard the second row as well. This reduces us to the following game:

$$P = \begin{bmatrix} -2 & 0 \\ 3 & -1 \end{bmatrix}$$

In this matrix, neither row dominates the other and neither column dominates the other. So, this is as far as we can go with this line of argument. We call this **reduction by dominance**.

Reduction by Dominance

One *row* **dominates** another if every entry in the former is greater than or equal to the corresponding entry in the latter. Put another way, one row dominates another if it is always at least as good for the row player.

One *column* dominates another if every entry in the former is less than or equal to the corresponding entry in the latter. Put another way, one column dominates another if it is always at least as good for the column player.

Procedure for Reducing by Dominance:

1. Check whether there is any row in the (remaining) matrix that is dominated by another row. Remove all dominated rows.
2. Check whether there is any column in the (remaining) matrix that is dominated by another column. Remove all dominated columns.
3. Repeat steps 1 and 2 until there are no dominated rows or columns.

Let us now go back to the “television ratings wars” example and see how we can solve a game using the minimax criterion once we are down to a 2×2 payoff matrix.

Example 3 Solving a 2×2 Game

Continuing the preceding example:

- a. Find the optimal strategy for RTV.
- b. Find the optimal strategy for CTV.
- c. Find the expected value of the game if RTV and CTV use their optimal strategies.

Solution As in the text, we begin by reducing the game by dominance, which brings us down to the following 2×2 game:

		CTV	
		Nature Doc.	Ballet
RTV	Reality Show	−2	0
	Movie	3	−1

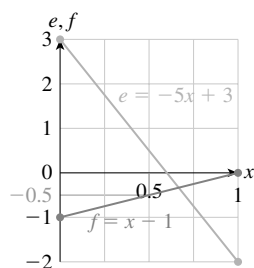


Figure 1

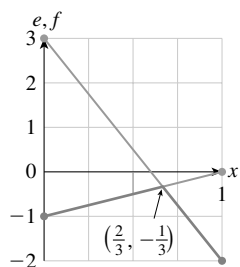


Figure 2

- a. Now let's find RTV's optimal strategy. Because we don't yet know what it is, we write down a general strategy:

$$R = [x \quad y]$$

Because $x + y = 1$, we can replace y by $1 - x$:

$$R = [x \quad 1 - x]$$

We know that CTV's best counterstrategy to R will be a pure strategy (see the discussion after Example 2), so let's compute the expected value of the game for each of CTV's possible pure strategies:

$$\begin{aligned} e &= [x \quad 1 - x] \begin{bmatrix} -2 & 0 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \\ &= (-2)x + 3(1 - x) = -5x + 3 \end{aligned}$$

$$\begin{aligned} f &= [x \quad 1 - x] \begin{bmatrix} -2 & 0 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \\ &= 0x - (1 - x) = x - 1 \end{aligned}$$

Because both e and f depend on x , we can graph them as in Figure 1.

If, for instance, RTV happened to choose $x = 0.5$, then the expected values for CTV's two pure strategies are $e = -5(1/2) + 3 = 1/2$ and $f = 1/2 - 1 = -1/2$. The worst outcome for RTV is the lower of the two, f , and this will be true wherever the graph of f is below the graph of e . On the other hand, if RTV chose $x = 1$, the graph of e would be lower and the worst possible expected value would be $e = -5(1) + 3 = -2$. Since RTV can choose x to be any value between 0 and 1, the worst possible outcomes are those shown by the orange portion of the graph in Figure 2.

Because RTV is trying to make the worst possible outcome as large as possible (that is, to minimize damages), it is seeking the point on the orange portion of the graph that is highest. This is the intersection point of the two lines. To calculate its coordinates, it's easiest to equate the two functions of x :

$$-5x + 3 = x - 1,$$

$$-6x = -4,$$

or
$$x = \frac{2}{3}$$

The e (or f) coordinate is then obtained by substituting $x = 2/3$ into the expression for e (or f) giving:

$$\begin{aligned} e &= -5 \left(\frac{2}{3} \right) + 3 \\ &= -\frac{1}{3} \end{aligned}$$

We conclude that RTV's best strategy is to take $x = 2/3$, giving an expected value of $-1/3$. In other words, RTV's optimal mixed strategy is:

$$R = \left[\frac{2}{3} \quad \frac{1}{3} \right]$$

Going back to the original game, RTV should show reality shows $2/3$ of the time and movies $1/3$ of the time. It should not bother showing any sitcoms or docudramas. It expects to lose, on average, 333 viewers to CTV, but all of its other options are worse.

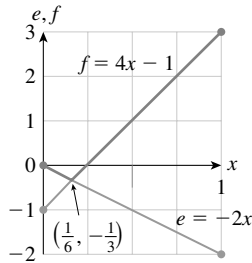


Figure 3

- b. To find CTV's optimal strategy, we must reverse roles and start by writing its unknown strategy as follows.

$$C = \begin{bmatrix} x \\ 1 - x \end{bmatrix}$$

We calculate the expected values for the two pure row strategies:

$$\begin{aligned} e &= [1 \quad 0] \begin{bmatrix} -2 & 0 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x \\ 1 - x \end{bmatrix} \\ &= -2x \end{aligned}$$

and

$$\begin{aligned} f &= [0 \quad 1] \begin{bmatrix} -2 & 0 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} x \\ 1 - x \end{bmatrix} \\ &= 3x - (1 - x) \\ &= 4x - 1 \end{aligned}$$

As with the row player, we know that the column player's best strategy will correspond to the intersection of the graphs of e and f (Figure 3).

(Why is the upper edge orange, rather than the lower edge?) The graphs intersect when

$$-2x = 4x - 1$$

or

$$x = \frac{1}{6}$$

The corresponding value of e (or f) is

$$e = -2 \left(\frac{1}{6} \right) = -\frac{1}{3}$$

Thus CTV's optimal mixed strategy is $\left[\frac{1}{6} \quad \frac{5}{6} \right]^T$ and the expected value is $-1/3$. So, CTV should show nature documentaries $1/6$ of the time and ballets $5/6$ of the time. It should not bother to show symphonies or operas. It expects to gain, on average, 333 viewers from RTV.

- c. We can now calculate the expected value of the game as usual, using the optimal strategies we found in parts (a) and (b).

$$\begin{aligned} e &= RPC \\ &= \begin{bmatrix} 2 & 1 \\ 3 & 3 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{6} \\ \frac{5}{6} \end{bmatrix} \\ &= -\frac{1}{3} \end{aligned}$$

† *Before we go on...* In Example 3, it is no accident that the expected value with the optimal strategies equals the expected values we found in (a) and (b). If we call the expected value with the optimal strategies the **expected value of the game**, the row player's optimal strategy guarantees an average payoff no smaller than the expected value, while the column player's optimal strategy guarantees an average payoff no larger. Together they force the average value to be the expected value of the game. ■

Expected Value of a Game

The **expected value of a game** is its expected value when the row and column players use their optimal (minimax) strategies. By using its optimal strategy, the row player guarantees an expected value no lower than the expected value of the game, no matter what the column player does. Similarly, by using its optimal strategy, the column player guarantees an expected value no higher than the expected value of the game, no matter what the row player does.

Q: What about games that don't reduce to 2×2 matrices? Are these solved in a similar way?

A: The method illustrated in the above example cannot easily be generalized to solve bigger games (i.e., games that cannot be reduced to 2×2 matrices); solving even a 2×3 game using this approach would require us to consider graphs in three dimensions. To be able to solve games of arbitrary size, we need to wait until the next chapter (Section 4.5) where we describe a method for solving a game, using the simplex method, that works for all payoff matrices. ■

Strictly Determined Games

The following example illustrates an interesting special case.

Example 4 Strictly Determined Game

Solve the following game:

$$\mathbf{A} \begin{matrix} & \mathbf{B} \\ & p & q & r \\ s & \begin{bmatrix} -4 & -3 & 3 \end{bmatrix} \\ t & \begin{bmatrix} 2 & -1 & -2 \end{bmatrix} \\ u & \begin{bmatrix} 1 & 0 & 2 \end{bmatrix} \end{matrix}$$

Solution If we look carefully at this matrix, we see that no row dominates another and no column dominates another, so we can't reduce it. Nor do we know how to solve a 3×3 game, so it looks as if we're stuck. However, there is a way to understand this particular game. With the minimax criterion in mind, let's begin by considering the worst possible outcomes for the row player for each possible move. We do this by circling the smallest payoff in each row, the **row minima**:

$$\mathbf{A} \begin{matrix} & \mathbf{B} \\ & p & q & r & \text{Row minima} \\ s & \begin{bmatrix} \ominus 4 & -3 & 3 \end{bmatrix} & -4 \\ t & \begin{bmatrix} 2 & -1 & \ominus 2 \end{bmatrix} & -2 \\ u & \begin{bmatrix} 1 & \ominus 0 & 2 \end{bmatrix} & 0 \leftarrow \text{largest} \end{matrix}$$

So, for example, if A plays move s , the worst possible outcome is to lose 4. Player A takes the least risk by using move u , which has the largest row minimum.

We do the same thing for the column player, remembering that smaller payoffs are better for B and larger payoffs worse. We draw a box around the largest payoff in each column, the **column maxima**:

		B		
		<i>p</i>	<i>q</i>	<i>r</i>
A	<i>s</i>	−4	−3	3
	<i>t</i>	2	−1	−2
	<i>u</i>	1	0	2
Column maxima		2	0	3
			↑	
			smallest	

Player B takes the least risk by using move *q*, which has the smallest column maximum.

Now put the circles and boxes together:

		B		
		<i>p</i>	<i>q</i>	<i>r</i>
A	<i>s</i>	−4	−3	3
	<i>t</i>	2	−1	−2
	<i>u</i>	1	0	2

Notice that the *uq* entry is both circled and boxed: it is both a row minimum and a column maximum. We call such an entry a **saddle point**.

Now we claim that the optimal strategy for A is to always play *u* while the optimal strategy for B is to always play *q*. By playing *u*, A guarantees that the payoff will be 0 or higher, no matter what B does, so the expected value of the game has to be at least 0. On the other hand, by playing *q*, B guarantees that the payoff will be 0 or less, so the expected value of the game has to be no more than 0. That means that the expected value of the game must be 0, and A and B have no strategies that could do any better for them than the pure strategies *u* and *q*.

† *Before we go on...* You should consider what happens in an example like the television rating wars game of Example 3. In that game, the largest row minimum is -1 while the smallest column maximum is 0; there is no saddle point. The row player can force a payoff of at least -1 by playing a pure strategy (always showing movies, for example), but can do better, forcing an average payoff of $-1/3$, by playing a mixed strategy. Similarly, the column player can force the payoff to be 0 or less with a pure strategy, but can do better, forcing an average payoff of $-1/3$, with a mixed strategy. Only when there is a saddle point will pure strategies be optimal. ■

Strictly Determined Game

A **saddle point** is a payoff that is simultaneously a row minimum and a column maximum (both boxed and circled in our approach). If a game has a saddle point, the corresponding row and column strategies are the optimal ones, and we say that the game is **strictly determined**.

FAQs Solving a Game



using Technology

To play a strictly determined game against your computer, and also to automatically reduce by dominance and solve arbitrary games up to 5×5 , go online and follow the path

Chapter 3

→ Tools

→ Game Theory Tool

Q: We've seen several ways of trying to solve a game. What should I do and in what order?

A: Here are the steps you should take when trying to solve a game:

1. Reduce by dominance. This should always be your first step.
2. If you were able to reduce to a 1×1 game, you're done. The optimal strategies are the corresponding pure strategies, as they dominate all the others.
3. Look for a saddle point in the reduced game. If it has one, the game is strictly determined, and the corresponding pure strategies are optimal.
4. If your reduced game is 2×2 and has no saddle point, use the method of Example 3 to find the optimal mixed strategies.
5. If your reduced game is larger than 2×2 and has no saddle point, you have to use linear programming to solve it, but that will have to wait until the following chapter. ■

3.4 EXERCISES

● denotes basic skills exercises

◆ denotes challenging exercises

tech Ex indicates exercises that should be solved using technology

Reduce the payoff matrices in Exercises 1–6 by dominance.

1. ●
$$\begin{array}{c} \mathbf{A} \\ \mathbf{B} \end{array} \begin{array}{ccc} p & q & r \\ \begin{bmatrix} 1 & 1 & 10 \\ 2 & 3 & -4 \end{bmatrix} \end{array}$$

2. ●
$$\begin{array}{c} \mathbf{A} \\ \mathbf{B} \end{array} \begin{array}{ccc} p & q & r \\ \begin{bmatrix} 2 & 0 & 10 \\ 15 & -4 & -5 \end{bmatrix} \end{array}$$

3. ●
$$\begin{array}{c} \mathbf{A} \\ \mathbf{B} \end{array} \begin{array}{ccc} a & b & c \\ \begin{bmatrix} 1 & 2 & -4 & -9 \\ -1 & -2 & -3 \\ 3 & 5 & 0 & -1 \end{bmatrix} \end{array}$$

4. ●
$$\begin{array}{c} \mathbf{A} \\ \mathbf{B} \end{array} \begin{array}{ccc} a & b & c \\ \begin{bmatrix} 1 & 0 & -1 & -5 \\ -3 & -10 & 10 \\ 3 & 2 & 3 & -4 \end{bmatrix} \end{array}$$

5. ●
$$\begin{array}{c} \mathbf{A} \\ \mathbf{B} \end{array} \begin{array}{ccc} a & b & c \\ \begin{bmatrix} p & 1 & -1 & -5 \\ q & 4 & 0 & 2 \\ r & 3 & -3 & 10 \\ s & 3 & -5 & -4 \end{bmatrix} \end{array}$$

6. ●
$$\begin{array}{c} \mathbf{A} \\ \mathbf{B} \end{array} \begin{array}{ccc} a & b & c \\ \begin{bmatrix} p & 2 & -4 & 9 \\ q & 1 & 1 & 0 \\ r & -1 & -2 & -3 \\ s & 1 & 1 & -1 \end{bmatrix} \end{array}$$

In Exercises 7–12, decide whether the game is strictly determined. If it is, give the players' optimal pure strategies and the value of the game. *hint* [see Example 4]

7. ●
$$\begin{array}{c} \mathbf{A} \\ \mathbf{B} \end{array} \begin{array}{cc} p & q \\ \begin{bmatrix} 1 & 1 \\ 2 & -4 \end{bmatrix} \end{array}$$

8. ●
$$\begin{array}{c} \mathbf{A} \\ \mathbf{B} \end{array} \begin{array}{cc} p & q \\ \begin{bmatrix} -1 & 2 \\ 10 & -1 \end{bmatrix} \end{array}$$

9. ●
$$\begin{array}{c} \mathbf{A} \\ \mathbf{B} \end{array} \begin{array}{ccc} p & q & r \\ \begin{bmatrix} 2 & 0 & -2 \\ -1 & 3 & 0 \end{bmatrix} \end{array}$$

10. ●
$$\begin{array}{c} \mathbf{A} \\ \mathbf{B} \end{array} \begin{array}{ccc} p & q & r \\ \begin{bmatrix} -2 & 1 & -3 \\ -2 & 3 & -2 \end{bmatrix} \end{array}$$

11. ●
$$\begin{array}{c} \mathbf{A} \\ \mathbf{B} \end{array} \begin{array}{ccc} a & b & c \\ \begin{bmatrix} P & 1 & -1 & -5 \\ Q & 4 & -4 & 2 \\ R & 3 & -3 & -10 \\ S & 5 & -5 & -4 \end{bmatrix} \end{array}$$

12. ●
$$\begin{array}{c} \mathbf{A} \\ \mathbf{B} \end{array} \begin{array}{ccc} a & b & c \\ \begin{bmatrix} P & -2 & -4 & 9 \\ Q & 1 & 1 & 0 \\ R & -1 & -2 & -3 \\ S & 1 & 1 & -1 \end{bmatrix} \end{array}$$

In Exercises 13–16, calculate the expected value of the game with payoff matrix

$$P = \begin{bmatrix} 2 & 0 & -1 & 2 \\ -1 & 0 & 0 & -2 \\ -2 & 0 & 0 & 1 \\ 3 & 1 & -1 & 1 \end{bmatrix}$$

using the mixed strategies supplied. *hint* [see Example 1]

● basic skills ◆ challenging **tech** Ex technology exercise

13. ● $R = [0 \ 1 \ 0 \ 0]$, $C = [1 \ 0 \ 0 \ 0]^T$
 14. ● $R = [0 \ 0 \ 0 \ 1]$, $C = [0 \ 1 \ 0 \ 0]^T$
 15. ● $R = [0.5 \ 0.5 \ 0 \ 0]$, $C = [0 \ 0 \ 0.5 \ 0.5]^T$
 16. ● $R = [0 \ 0.5 \ 0 \ 0.5]$, $C = [0.5 \ 0.5 \ 0 \ 0]^T$

In Exercises 17–20, either a mixed column or row strategy is given. In each case, use

$$P = \begin{bmatrix} 0 & -1 & 5 \\ 2 & -2 & 4 \\ 0 & 3 & 0 \\ 1 & 0 & -5 \end{bmatrix}$$

and find the optimal pure strategy (or strategies) the other player should use. Express the answer as a row or column matrix. Also determine the resulting expected value of the game.

hint [see Example 2]

17. ● $C = [0.25 \ 0.75 \ 0]^T$ 18. ● $C = \left[\frac{1}{3} \ \frac{1}{3} \ \frac{1}{3} \right]^T$
 19. ● $R = \left[\frac{1}{2} \ 0 \ \frac{1}{4} \ \frac{1}{4} \right]$ 20. ● $R = [0.8 \ 0.2 \ 0 \ 0]$

In Exercises 21–24, find: **a.** the optimal mixed row strategy; **b.** the optimal mixed column strategy, and **c.** the expected value of the game in the event that each player uses his or her optimal mixed strategy. hint [see Example 3]

21. ● $P = \begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix}$ 22. ● $P = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}$
 23. ● $P = \begin{bmatrix} -1 & -2 \\ -2 & 1 \end{bmatrix}$ 24. ● $P = \begin{bmatrix} -2 & -1 \\ -1 & -3 \end{bmatrix}$

Applications

Set up the payoff matrix in each of Exercises 25–32.

25. ● **Games to Pass the Time** You and your friend have come up with the following simple game to pass the time: at each round, you simultaneously call “heads” or “tails.” If you have both called the same thing, your friend wins one point; if your calls differ, you win one point.
26. ● **Games to Pass the Time** Bored with the game in Exercise 25, you decide to use the following variation instead: If you both call “heads” your friend wins two points; if you both call “tails” your friend wins 1 point; if your calls differ, then you win two points if you called “heads,” and one point if you called “tails.”
27. ● **War Games** You are deciding whether to invade France, Sweden or Norway, and your opponent is simultaneously deciding which of these three countries to defend. If you invade a country that your opponent is defending, you will be defeated (payoff: -1), but if you invade a country your opponent is not defending, you will be successful (payoff: $+1$).
28. ● **War Games** You must decide whether to attack your opponent by sea or air, and your opponent must simultaneously decide whether to mount an all-out air defense, an all-out coastal defense (against an attack from the sea) or a combined air and coastal defense. If there is no defense for your mode of attack, you win 100 points. If your attack is met by a shared air and coastal defense, you win 50 points. If your attack is met by an all-out defense, you lose 200 points.
29. **Marketing** Your fast-food outlet, Burger Queen, has obtained a license to open branches in three closely situated South African cities: Brakpan, Nigel, and Springs. Your market surveys show that Brakpan and Nigel each provide a potential market of 2000 burgers a day, while Springs provides a potential market of 1000 burgers per day. Your company can only finance an outlet in one of those cities at the present time. Your main competitor, Burger Princess, has also obtained licenses for these cities, and is similarly planning to open only one outlet. If you both happen to locate at the same city, you will share the total business from all three cities equally, but if you locate in different cities, you will each get all the business in the cities in which you have located, plus half the business in the third city. The payoff is the number of burgers you will sell per day minus the number of burgers your competitor will sell per day.
30. **Marketing** Repeat Exercise 29, given that the potential sales markets in the three cities are: Brakpan, 2500 per day; Nigel, 1500 per day; Springs, 1200 per day.
31. **Betting** When you bet on a racehorse with odds of $m-n$, you stand to win m dollars for every bet of n dollars if your horse wins; for instance, if the horse you bet is running at 5–2 and wins, you will win \$5 for every \$2 you bet. (Thus a \$2 bet will return \$7.) Here are some actual odds from a 1992 race at Belmont Park, NY.²⁴ The favorite at 5–2 was Pleasant Tap. The second choice was Thunder Rumble at 7–2, while the third choice was Strike the Gold at 4–1. Assume you are making a \$10 bet on one of these horses. The payoffs are your winnings. (If your horse does not win, you lose your entire bet. Of course, it is possible for none of your horses to win.)
32. **Betting** Referring to Exercise 31, suppose that just before the race, there has been frantic betting on Thunder Rumble, with the result that the odds have dropped to 2–5. The odds on the other two horses remain unchanged.
33. ● **Price Wars** Computer Electronics, Inc. (CE) and the Gigantic Computer Store (GCS) are planning to discount the price they charge for the HAL Laptop Computer, of which they are the only distributors. Because Computer Electronics provides a free warranty service, they can generally afford to charge more. A market survey provides the following data on the

²⁴ SOURCE: *The New York Times*, September 18, 1992, p.B14.

gains to CE's market share that will result from different pricing decisions:

		GCS		
		\$900	\$1000	\$1200
CE	\$1000	15%	60%	80%
	\$1200	15%	60%	60%
	\$1300	10%	20%	40%

- Use reduction by dominance to determine how much each company should charge. What is the effect on CE's market share?
 - CE is aware that GCS is planning to use reduction of dominance to determine its pricing policy, and wants its market share to be as large as possible. What effect, if any, would the information about GCS have on CE's best strategy?
34. ● **More Price Wars** (Refer to Exercise 33) A new market survey results in the following revised data:

		GCS		
		\$900	\$1000	\$1200
CE	\$1000	20%	60%	60%
	\$1200	15%	60%	60%
	\$1300	10%	20%	40%

- Use reduction by dominance to determine how much each company should charge. What is the effect on CE's market share?
 - In general, why do price wars tend to force prices down?
35. ● **Wrestling Tournaments** City Community College (CCC) plans to host Midtown Military Academy (MMA) for a wrestling tournament. Each school has three wrestlers in the 190 lb. weight class: CCC has Pablo, Sal, and Edison, while MMA has Carlos, Marcus and Noto. Pablo can beat Carlos and Marcus, Marcus can beat Edison and Sal, Noto can beat Edison, while the other combinations will result in an even match. Set up a payoff matrix, and use reduction by dominance to decide which wrestler each team should choose as their champion. Does one school have an advantage over the other?
36. ● **Wrestling Tournaments** (Refer to Exercise 35) One day before the wrestling tournament discussed in Exercise 35, Pablo sustains a hamstring injury, and is replaced by Hans, who (unfortunately for CCC) can be beaten by both Carlos and Marcus. Set up the payoff matrix, and use reduction by dominance to decide which wrestler each team should choose as their champion. Does one school have an advantage over the other?
37. ● **The Battle of Rabaul-Lae**²⁵ In the Second World War, during the struggle for New Guinea, intelligence reports revealed

²⁵ As discussed in *Games and Decisions* by R.D. Luce and H. Raiffa Section 11.3 (New York; Wiley, 1957). This is based on an article in the *Journal of the Operations Research Society of America* 2 (1954) 365–385.

that the Japanese were planning to move a troop and supply convoy from the port of Rabaul at the Eastern tip of New Britain to Lae, which lies just west of New Britain on New Guinea. It could either travel via a northern route which was plagued by poor visibility, or by a southern route, where the visibility was clear. General Kenney, who was the commander of the Allied Air Forces in the area, had the choice of concentrating reconnaissance aircraft on one route or the other, and bombing the Japanese convoy once it was sighted. Kenney's

Japanese
Commander's Strategies

		Northern Route	Southern Route
Kenney's Strategies	Northern Route	2	2
	Southern Route	1	3

staff drafted the following outcomes for his choices, where the payoffs are estimated days of bombing time:

What would you have recommended to General Kenney? What would you have recommended to the Japanese Commander?²⁶ How much bombing time results if these recommendations are followed?

38. ● **The Battle of Rabaul-Lae** Referring to Exercise 37, suppose that General Kenney had a third alternative: Splitting his reconnaissance aircraft between the two routes, and resulting in the following estimates:

Japanese
Commander's Strategies

		Northern Route	Southern Route
Kenney's Strategies	Northern Route	2	2
	Split Reconnaissance	1.5	2.5
	Southern Route	1	3

What would you have recommended to General Kenney? What would you have recommended to the Japanese Commander? How much bombing time results if these recommendations are followed?

39. **The Prisoner's Dilemma** Slim Lefty and Joe Rap have been arrested for grand theft auto, having been caught red-handed driving away in a stolen 2005 Porsche. Although the police have more than enough evidence to convict them both, they feel that a confession would simplify the work of the prosecution. They decide to interrogate the prisoners separately. Slim and Joe are both told of the following plea-bargaining arrangement: if both confess, they will each receive a two-year

²⁶ The correct answers to parts (a) and (b) correspond to the actual decisions both commanders made.

sentence; if neither confesses, they will both receive five-year sentences, and if only one confesses (and thus squeals on the other), he will receive a suspended sentence, while the other will receive a 10-year sentence. What should Slim do?

- 40. More Prisoners' Dilemmas** Jane Good and Prudence Brown have been arrested for robbery, but the police lack sufficient evidence for a conviction, and so decide to interrogate them separately in the hope of extracting a confession. Both Jane and Prudence are told the following: if they both confess, they will each receive a five-year sentence; if neither confesses, they will be released; if one confesses, she will receive a suspended sentence, while the other will receive a ten-year sentence. What should Jane do?
- 41. Campaign Strategies**²⁷ Florida and Ohio are “swing states” that have a large bounty of electoral votes and are therefore highly valued by presidential campaign strategists. Suppose it is now the weekend before Election Day 2004, and each candidate (Bush and Kerry) can visit only one more state. Further, to win the election, Bush needs to win both of these states. Currently Bush has a 40% chance of winning Ohio and a 60% chance of winning Florida. Therefore, he has a $0.40 \times 0.60 = 0.24$, or 24% chance of winning the election. Assume that each candidate can increase his probability of winning a state by 10% if he, and not his opponent, visits that state. If both candidates visit the same state, there is no effect.
- Set up a payoff matrix with Bush as the row player and Kerry as the column player, where the payoff for a specific set of circumstances is the probability (expressed as a percentage) that Bush will win both states.
 - Where should each candidate visit under the circumstances?
- 42. Campaign Strategies** Repeat Exercise 41, this time assuming that Bush has an 80% chance of winning Ohio and a 90% chance of winning Florida.
- 43. Retail Discount Wars** Just one week after your Abercrom B men's fashion outlet has opened at a new location opposite Burger King[®] in the Mall, your rival, Abercrom A, opens up directly across from you. You have been informed that Abercrom A is about to launch either a 30% off everything sale or a 50% off everything sale. You, on the other hand, have decided to either *increase* prices (to make your store seem more exclusive) or do absolutely nothing. You construct the following payoff matrix, where the payoffs represent the number of customers your outlet can expect to gain from Abercrom A:

		Abercrom A	
		30% Off	50% Off
Abercrom B	Do Nothing	-60	-40
	Increase prices	30	-50

There is a 20% chance that Abercrom A will opt for the “30% off” sale and an 80% chance that they will opt for the “50% Off” sale. Your sense from upper management at Abercrom B is that there is a 50% chance you will be given the go-ahead to raise prices. What is the expected resulting effect on your customer base?

- 44. More Retail Discount Wars** Your Abercrom B mens fashion outlet has a 30% chance of launching an expensive new line of used auto-mechanic dungarees (complete with grease stains) and a 70% chance of staying instead with its traditional torn military-style dungarees. Your rival across from you in the mall, Abercrom A, appears to be deciding between a line of torn gym shirts and a more daring line of “empty shirts” (that is, empty shirt boxes). Your corporate spies reveal that there is a 20% chance that Abercrom A will opt for the empty shirt option. The following payoff matrix gives the number of customers your outlet can expect to gain from Abercrom A in each situation:

		Abercrom A	
		Torn Shirts	Empty Shirts
Abercrom B	Mechanics	10	-40
	Military	-30	50

There is a 20% chance that Abercrom A will opt for the “30% off” sale and an 80% chance that they will opt for the “50% Off” sale. Your sense from upper management at Abercrom B is that there is a 50% chance you will be given the go-ahead to raise prices. What is the expected resulting effect on your customer base?

- 45. Factory Location**²⁸ A manufacturer of electrical machinery is located in a cramped, though low-rent, factory close to the center of a large city. The firm needs to expand, and it could do so in one of three ways: (1) remain where it is and install new equipment, (2) move to a suburban site in the same city, or (3) relocate in a different part of the country where labor is cheaper. Its decision will be influenced by the fact that one of the following will happen: (I) the government may introduce a program of equipment grants, (II) a new suburban highway may be built, or (III) the government may institute a policy of financial help to companies who move into regions of high unemployment. The value to the company of each combination is given in the following payoff matrix.

		Government's Options		
		I	II	III
Manufacturer's Options	1	200	150	140
	2	130	220	130
	3	110	110	220

²⁷ Based on *Game Theory for Swingers: What states should the candidates visit before Election Day?* by Jordan Ellenberg. SOURCE: www.slate.com.

²⁸ Adapted from an example in *Location in Space: Theoretical Perspectives in Economic Geography* by P. Dicken and P.E. Lloyd, Harper & Row, 1990.

If the manufacturer judges that there is a 20% probability that the government will go with option I, a 50% probability that they will go with option II, and a 30% probability that they will go with option III, what is the manufacturer's best option?

46. ● **Crop Choice**²⁹ A farmer has a choice of growing wheat, barley, or rice. Her success will depend on the weather, which could be dry, average, or wet. Her payoff matrix is as follows.

		Weather		
		Dry	Average	Wet
Crop Choices	Wheat	20	20	10
	Barley	10	15	20
	Rice	10	20	20

If the probability that the weather will be dry is 10%, the probability that it will be average is 60%, and the probability that it will be wet is 30%, what is the farmer's best choice of crop?

47. ● **Study Techniques** Your mathematics test is tomorrow, and will cover the following topics: game theory, linear programming, and matrix algebra. You have decided to do an "all-nighter" and must determine how to allocate your eight hours of study time among the three topics. If you were to spend the entire eight hours on any one of these topics (thus using a pure strategy) you feel confident that you would earn a 90% score on that portion of the test, but would not do so well on the other topics. You have come up with the following table, where the entries are your expected scores. (The fact that linear programming and matrix algebra are used in game theory is reflected in these numbers).

Your Strategies	Test		
	Game Theory	Linear Programming	Matrix Algebra
Game Theory	90	70	70
Linear Programming	40	90	40
Matrix Algebra	60	40	90

You have been told that the test will be weighted as follows: game theory: 25%; linear programming: 50%; matrix algebra: 25%.

- If you spend 25% of the night on game theory, 50% on linear programming, and 25% on matrix algebra, what score do you expect to get on the test?
- Is it possible to improve on this by altering your study schedule? If so, what is the highest score you can expect on the test?

- If your study schedule is according to part (a) and your teacher decides to forget her promises about how the test will be weighted and instead base it all on a single topic, which topic would be worst for you, and what score could you expect on the test?

48. ● **Study Techniques** Your friend Joe has been spending all of his time in fraternity activities, and thus knows absolutely nothing about any of the three topics on tomorrow's math test. (See Exercise 47.) Because you are recognized as an expert on the use of game theory to solve study problems, he has turned to you for advice as to how to spend his "all-nighter." As the following table shows, his situation is not so rosy. (Since he knows no linear programming or matrix algebra, the table shows, for instance, that studying game theory all night will not be much use in preparing him for this topic.)

Joe's Strategies	Test		
	Game Theory	Linear Programming	Matrix Algebra
Game Theory	30	0	20
Linear Programming	0	70	0
Matrix Algebra	0	0	70

Assuming that the test will be weighted as described in Exercise 47, what are the answers to parts (a), (b), and (c) as they apply to Joe?

49. **Staff Cutbacks** Frank Tempest manages a large snowplow service in Manhattan, Kansas, and is alarmed by the recent weather trends; there have been no significant snowfalls since 1993. He is therefore contemplating laying off some of his workers, but is unsure about whether to lay off 5, 10 or 15 of his 50 workers. Being very methodical, he estimates his annual net profits based on four possible annual snowfall figures: 0 inches, 20 inches, 40 inches and 60 inches. (He takes into account the fact that, if he is running a small operation in the face of a large annual snowfall, he will lose business to his competitors because he will be unable to discount on volume.)

	0 inches	20 inches	40 inches	60 inches
5 laid off	-\$500,000	-\$200,000	\$10,000	\$200,000
10 laid off	-\$200,000	\$0	\$0	\$0
15 laid off	-\$100,000	\$10,000	-\$200,000	-\$300,000

- During the past 10 years, the region has had 0 inches twice, 20 inches twice, 40 inches 3 times, and 60 inches 3 times. Based on this information, how many workers should Tempest lay off, and how much would it cost him?
- There is a 50% chance that Tempest will lay off 5 workers and a 50% chance that he will lay off 15 workers. What is the worst thing Nature can do to him in terms of snowfall? How much would it cost him?

²⁹ Ibid.

- c. The Gods of Chaos (who control the weather) know that Tempest is planning to use the strategy in part (a), and are determined to hurt Tempest as much as possible. Tempest, being somewhat paranoid, suspects it too. What should he do?

50. **Textbook Writing** You are writing a college-level textbook on finite mathematics, and are trying to come up with the best combination of word problems. Over the years, you have accumulated a collection of amusing problems, serious applications, long complicated problems, and “generic” problems.³⁰ Before your book is published, it must be scrutinized by several reviewers who, it seems, are never satisfied with the mix you use. You estimate that there are three kinds of reviewers: the “no-nonsense” types who prefer applications and generic problems, the “dead serious” types, who feel that a college-level text should be contain little or no humor and lots of long complicated problems, and the “laid-back” types, who believe that learning best takes place in a light-hearted atmosphere bordering on anarchy. You have drawn up the following chart, where the payoffs represent the reactions of reviewers on a scale of -10 (ballistic) to $+10$ (ecstatic):

		Reviewers		
		No-Nonsense	Dead Serious	Laid-Back
You	Amusing	-5	-10	10
	Serious	5	3	0
	Long	-5	5	3
	Generic	5	3	-10

- a. Your first draft of the book contained no generic problems, and equal numbers of the other categories. If half the reviewers of your book were “dead serious” and the rest were equally divided between the “no-nonsense” and “laid-back” types, what score would you expect?
- b. In your second draft of the book, you tried to balance the content by including some generic problems and eliminating several amusing ones, and wound up with a mix of which one eighth were amusing, one quarter were serious, three eighths were long, and a quarter were generic. What kind of reviewer would be *least* impressed by this mix?
- c. What kind of reviewer would be *most* impressed by the mix in your second draft?
51. ● **Advertising** The Softex Shampoo Company is considering how to split its advertising budget between ads on two radio stations: WISH and WASH. Its main competitor, Splish Shampoo, Inc. has found out about this, and is considering countering Softex’s ads with its own, on the same radio stations. (Proposed jingle: *Softex, Shmofitex; Splash with Splish*) Softex has calculated that, were it to devote its entire adver-

tising budget to ads on WISH, it would increase revenues in the coming month by \$100,000 in the event that Splish was running all its ads on the less popular WASH, but would lose \$20,000 in revenues if Splish ran its ads on WISH. If, on the other hand, it devoted its entire budget to WASH ads, it would neither increase nor decrease revenues in the event that Splish was running all its ads on the more popular WISH, and would in fact lose \$20,000 in revenues if Splish ran its ads on WASH. What should Softex do, and what effect will this have on revenues?

52. ● **Labor Negotiations** The management team of the Abstract Concrete Company is negotiating a three-year contract with the labor unions at one of its plants, and is trying to decide on its offer for a salary increase. If it offers a 5% increase and the unions accept the offer, Abstract Concrete will gain \$20 million in projected profits in the coming year, but if labor rejects the offer, the management team predicts that it will be forced to increase the offer to the union demand of 15%, thus halving the projected profits. If Abstract Concrete offers a 15% increase, the company will earn \$10 million in profits over the coming year if the unions accept. If the unions reject, they will probably go out on strike (because management has set 15% as its upper limit) and management has decided that it can then in fact gain \$12 million in profits by selling out the defunct plant in retaliation. What intermediate percentage should the company offer, and what profit should it project?

Communication and Reasoning Exercises

53. ● Why is a saddle point called a “saddle point”?
54. ● Can the payoff in a saddle point ever be larger than all other payoffs in a game? Explain.
55. ◆ One day, while browsing through an old *Statistical Abstract of the United States*, you came across the following data which show the number of females employed (in thousands) in various categories according to their educational attainment.³¹

	Managerial/ Professional	Technical/Sales/ Administrative	Service	Precision Production	Operators/ Fabricators
Less than 4 years high school	260	1080	2020	260	1400
4 years of high school only	2430	9510	3600	570	2130
1 to 3 years of college	2690	5080	1080	160	350
At least 4 years of college	7210	2760	380	70	110

³⁰ of the following type: “An oil company has three refineries: A, B and C, each of which uses three processes: P_1 , P_2 , and P_3 . Process P_1 uses 100 units of chemical C_1 and costs \$100 per day . . .”

³¹ SOURCE: *Statistical Abstract of the United States 1991* (111th Ed.) U.S. Department of Commerce, Economics and Statistics Administration, and Bureau of the Census.

Because you had been studying game theory that day, the first thing you did was to search for a saddle point. Having found one, you conclude that, as a female, your best strategy in the job market is to forget about a college career. Find the flaw in this reasoning.

56. ♦ Exercises 47 and 48 seem to suggest that studying a single topic prior to an exam is better than studying all the topics in that exam. Comment on this discrepancy between the game theory result and common sense.

57. ♦ Explain what is wrong with a decision to play the mixed strategy $[0.5 \ 0.5]$ by alternating the two strategies: play the first strategy on the odd-numbered moves and the second strategy on the even-numbered moves. Illustrate your argument by devising a game in which your best strategy is $[0.5 \ 0.5]$.

58. ♦ Describe a situation in which both a mixed strategy and a pure strategy are equally effective.

● basic skills ♦ challenging **tech**Ex technology exercise

3.5 Input-Output Models

In this section we look at an application of matrix algebra developed by Wassily Leontief (1906–1999) in the middle of the twentieth century. In 1973, he won the Nobel Prize in Economics for this work. The application involves analyzing national and regional economies by looking at how various parts of the economy interrelate. We'll work out some of the details by looking at a simple scenario.

First, we can think of the economy of a country or a region as being composed of various **sectors**, or groups of one or more industries. Typical sectors are the manufacturing sector, the utilities sector, and the agricultural sector. To introduce the basic concepts, we shall consider two specific sectors: the coal-mining sector (Sector 1) and the electric utilities sector (Sector 2). Both produce a commodity: the coal-mining sector produces coal, and the electric utilities sector produces electricity. We measure these products by their dollar value. By **one unit** of a product, we mean \$1 worth of that product.

Here is the scenario.

1. To produce one unit (\$1 worth) of coal, assume that the coal-mining sector uses 50¢ worth of coal (to power mining machinery, say) and 10¢ worth of electricity.
2. To produce one unit (\$1 worth) of electricity, assume that the electric utilities sector uses 25¢ worth of coal and 25¢ worth of electricity.

These are *internal* usage figures. In addition to this, assume that there is an *external* demand (from the rest of the economy) of 7000 units (\$7,000 worth) of coal and 14,000 units (\$14,000 worth) of electricity over a specific time period (one year, say). Our basic question is: How much should each of the two sectors supply in order to meet both internal and external demand?

The key to answering this question is to set up equations of the form:

$$\text{Total supply} = \text{Total demand}$$

The unknowns, the values we are seeking, are

x_1 = the total supply (in units) from Sector 1 (coal) and

x_2 = the total supply (in units) from Sector 2 (electricity)

Our equations then take the following form:

Total supply from Sector 1 = Total demand for Sector 1 products

$$x_1 = \underset{\substack{\uparrow \\ \text{Coal required by Sector 1}}}{0.50x_1} + \underset{\substack{\uparrow \\ \text{Coal required by Sector 2}}}{0.25x_2} + \underset{\substack{\uparrow \\ \text{External demand for coal}}}{7000}$$

Total supply from Sector 2 = Total demand for Sector 2 products

$$x_2 = \underset{\substack{\uparrow \\ \text{Electricity required by Sector 1}}}{0.10x_1} + \underset{\substack{\uparrow \\ \text{Electricity required by Sector 2}}}{0.25x_2} + \underset{\substack{\uparrow \\ \text{External demand for electricity}}}{14,000}$$

This is a system of two linear equations in two unknowns:

$$x_1 = 0.50x_1 + 0.25x_2 + 7000$$

$$x_2 = 0.10x_1 + 0.25x_2 + 14,000$$

We can rewrite this system of equations in matrix form as follows:

$$\underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\text{Production}} = \underbrace{\begin{bmatrix} 0.50 & 0.25 \\ 0.10 & 0.25 \end{bmatrix}}_{\text{Internal demand}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\text{Production}} + \underbrace{\begin{bmatrix} 7000 \\ 14,000 \end{bmatrix}}_{\text{External demand}}$$

In symbols,

$$X = AX + D$$

Here,

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

is called the **production vector**. Its entries are the amounts produced by the two sectors. The matrix

$$D = \begin{bmatrix} 7000 \\ 14,000 \end{bmatrix}$$

is called the **external demand vector**, and

$$A = \begin{bmatrix} 0.50 & 0.25 \\ 0.10 & 0.25 \end{bmatrix}$$

is called the **technology matrix**. The entries of the technology matrix have the following meanings:

a_{11} = units of Sector 1 needed to produce one unit of Sector 1

a_{12} = units of Sector 1 needed to produce one unit of Sector 2

a_{21} = units of Sector 2 needed to produce one unit of Sector 1

a_{22} = units of Sector 2 needed to produce one unit of Sector 2

You can remember this order by the slogan “In the side, out the top.”

Now that we have the matrix equation

$$X = AX + D$$

we can solve it as follows. First, subtract AX from both sides:

$$X - AX = D$$

Because $X = IX$, where I is the 2×2 identity matrix, we can rewrite this as

$$IX - AX = D$$

Now factor out X :

$$(I - A)X = D$$

If we multiply both sides by the inverse of $(I - A)$, we get the solution

$$X = (I - A)^{-1}D$$

Input-Output Model

In an input-output model, an economy (or part of one) is divided into n **sectors**. We then record the $n \times n$ **technology matrix** A , whose ij th entry is the number of units from Sector i used in producing one unit from Sector j (“in the side, out the top”). To meet an **external demand** of D , the economy must produce X , where X is the **production vector**. These are related by the equations

$$X = AX + D$$

or

$$X = (I - A)^{-1}D \quad \text{Provided } (I - A) \text{ is invertible}$$

quick Example

In the scenario above, $A = \begin{bmatrix} 0.50 & 0.25 \\ 0.10 & 0.25 \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, and $D = \begin{bmatrix} 7,000 \\ 14,000 \end{bmatrix}$.

The solution is

$$\begin{aligned} X &= (I - A)^{-1}D \\ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} &= \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.50 & 0.25 \\ 0.10 & 0.25 \end{bmatrix} \right)^{-1} \begin{bmatrix} 7,000 \\ 14,000 \end{bmatrix} && \text{Calculate } I - A \\ &= \begin{bmatrix} 0.50 & -0.25 \\ -0.10 & 0.75 \end{bmatrix}^{-1} \begin{bmatrix} 7,000 \\ 14,000 \end{bmatrix} && \text{Calculate } (I - A)^{-1} \\ &= \begin{bmatrix} \frac{15}{7} & \frac{5}{7} \\ \frac{2}{7} & \frac{10}{7} \end{bmatrix} \begin{bmatrix} 7,000 \\ 14,000 \end{bmatrix} \\ &= \begin{bmatrix} 25,000 \\ 22,000 \end{bmatrix} \end{aligned}$$

In other words, to meet the demand, the economy must produce \$25,000 worth of coal and \$22,000 worth of electricity.

The next example uses actual data from the U.S. economy (we have rounded the figures to make the computations less complicated). It is rare to find input-output data already packaged for you as a technology matrix. Instead, the data commonly found in statistical sources come in the form of “input-output tables,” from which we will have to construct the technology matrix.



Dynamic Graphics, Inc./Jupiterimages

Example 1 Petroleum and Natural Gas

Consider two sectors of the U.S. economy: crude petroleum and natural gas (*crude*) and petroleum refining and related industries (*refining*). According to government figures,* in 1998 the crude sector used \$27,000 million worth of its own products and \$750 million worth of the products of the refining sector to produce \$87,000 million worth of goods (crude oil and natural gas). The refining sector in the same year used \$59,000 million worth of the products of the crude sector and \$15,000 million worth of its own products to produce \$140,000 million worth of goods (refined oil and the like). What was the technology matrix for these two sectors? What was left over from each of these sectors for use by other parts of the economy or for export?

Solution First, for convenience, we record the given data in the form of a table, called the **input-output table**. (All figures are in millions of dollars.)

		<i>To</i>	
		Crude	Refining
<i>From</i>	Crude	27,000	59,000
	Refining	750	15,000
Total Output		87,000	140,000

The entries in the top portion are arranged in the same way as those of the technology matrix: The ij th entry represents the number of units of Sector i that went to Sector j . Thus, for instance, the 59,000 million entry in the 1, 2 position represents the number of units of Sector 1, crude, that were used by Sector 2, refining. (“In the side, out the top”.)

We now construct the technology matrix. The technology matrix has entries a_{ij} = units of Sector i used to produce *one* unit of Sector j . Thus,

a_{11} = units of crude to produce one unit of crude. We are told that 27,000 million units of crude were used to produce 87,000 million units of crude. Thus, to produce *one* unit of crude, $27,000/87,000 \approx 0.31$ units of crude were used, and so $a_{11} \approx 0.31$. (We have rounded this value to two significant digits; further digits are not reliable due to rounding of the original data.)

a_{12} = units of crude to produce one unit of refined:

$$a_{12} = 59,000/140,000 \approx 0.42$$

a_{21} = units of refined to produce one unit of crude:

$$a_{21} = 750/87,000 \approx 0.0086$$

a_{22} = units of refined to produce one unit of refined:

$$a_{22} = 15,000/140,000 \approx 0.11$$

This gives the technology matrix

$$A = \begin{bmatrix} 0.31 & 0.42 \\ 0.0086 & 0.11 \end{bmatrix} \quad \text{Technology Matrix}$$

In short we obtained the technology matrix from the input-output table by dividing the Sector 1 column by the Sector 1 total, and the Sector 2 column by the Sector 2 total.

* The data have been rounded to two significant digits. SOURCE: *Survey of Current Business*, December, 2001, U.S. Department of Commerce. The *Survey of Current Business* and the input-output tables themselves are available at the website of the Department of Commerce’s Bureau of Economic Analysis, www.bea.gov.

Now we also know the total output from each sector, so *we have already been given the production vector*:

$$X = \begin{bmatrix} 87,000 \\ 140,000 \end{bmatrix} \quad \text{Production Vector}$$

What we are asked for is the external demand vector D , the amount available for the outside economy. To find D , we use the equation

$$X = AX + D \quad \text{Relationship of } X, A, \text{ and } D$$

where, this time, we are given A and X , and must solve for D . Solving for D gives

$$\begin{aligned} D &= X - AX \\ &= \begin{bmatrix} 87,000 \\ 140,000 \end{bmatrix} - \begin{bmatrix} 0.31 & 0.42 \\ 0.0086 & 0.11 \end{bmatrix} \begin{bmatrix} 87,000 \\ 140,000 \end{bmatrix} \\ &\approx \begin{bmatrix} 87,000 \\ 140,000 \end{bmatrix} - \begin{bmatrix} 86,000 \\ 16,000 \end{bmatrix} = \begin{bmatrix} 1000 \\ 124,000 \end{bmatrix} \quad \text{We rounded to 2 digits}^\dagger \end{aligned}$$

The first number, \$1000 million, is the amount produced by the crude sector that is available to be used by other parts of the economy or to be exported. (In fact, because something has to happen to all that crude petroleum and natural gas, this is the amount actually used or exported, where use can include stockpiling.) The second number, \$124,000 million, represents the amount produced by the refining sector that is available to be used by other parts of the economy or to be exported. Complete the example using a TI-83/84 or Excel.

Note that we could have calculated D more simply from the input-output table. The internal use of units from the crude sector was the sum of the outputs from that sector:

$$27,000 + 59,000 = 86,000$$

Because 87,000 units were actually produced by the sector, that left a surplus of $87,000 - 86,000 = 1000$ units for export. We could compute the surplus from the refining sector similarly. (The two calculations actually come out slightly different, because we rounded the intermediate results.) The calculation in Example 2 below cannot be done as trivially, however.

[†]Why?



using Technology

See the Technology Guides at the end of the chapter to see how to compute the technology matrix and then calculate the external demand vector using a TI 83/84 or Excel.

Input-Output Table

National economic data are often given in the form of an **input-output table**. The ij th entry in the top portion of the table is the number of units that go from Sector i to Sector j . The “Total outputs” are the total numbers of units produced by each sector. We obtain the technology matrix from the input-output table by dividing the Sector 1 column by the Sector 1 total, the Sector 2 column by the Sector 2 total, and so on.

quick Example

Input-Output Table:

		To	Skateboards	Wood
From	Skateboards		20,000*	0
	Wood		100,000	500,000
Total Output			200,000	5,000,000

* The production of skateboards required skateboards due to the fact that skateboard workers tend to commute to work on (what else?) skateboards!

Technology Matrix:

$$A = \begin{bmatrix} \frac{20,000}{200,000} & \frac{0}{5,000,000} \\ \frac{100,000}{200,000} & \frac{500,000}{5,000,000} \end{bmatrix} = \begin{bmatrix} 0.1 & 0 \\ 0.5 & 0.1 \end{bmatrix}$$

Example 2 Rising Demand

Suppose that external demand for refined petroleum rises to \$200,000 million, but the demand for crude remains \$1000 million (as in Example 1). How do the production levels of the two sectors considered in Example 1 have to change?

Solution We are being told that now

$$D = \begin{bmatrix} 1000 \\ 200,000 \end{bmatrix}$$

and we are asked to find X . Remember that we can calculate X from the formula

$$X = (I - A)^{-1}D$$

Now

$$I - A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 0.31 & 0.42 \\ 0.0086 & 0.11 \end{bmatrix} = \begin{bmatrix} 0.69 & -0.42 \\ -0.0086 & 0.89 \end{bmatrix}$$

We take the inverse using our favorite technique and find that, to four significant digits,*

$$(I - A)^{-1} \approx \begin{bmatrix} 1.458 & 0.6880 \\ 0.01409 & 1.130 \end{bmatrix}$$

Now we can compute X :

$$X = (I - A)^{-1}D = \begin{bmatrix} 1.458 & 0.6880 \\ 0.01409 & 1.130 \end{bmatrix} \begin{bmatrix} 1000 \\ 200,000 \end{bmatrix} \approx \begin{bmatrix} 140,000 \\ 230,000 \end{bmatrix}$$

(As in Example 1, we have rounded all the entries in the answer to two significant digits.) Comparing this vector to the production vector used in Example 1, we see that production in the crude sector has to increase from \$87,000 million to \$140,000 million, while production in the refining sector has to increase from \$140,000 million to \$230,000 million.

* Since A is accurate to two digits, we should use more than two significant digits in intermediate calculations so as not to lose additional accuracy. We must, of course, round the final answer to two digits.

Note Using the matrix $(I - A)^{-1}$, we have a slightly different way of solving Example 2. We are asking for the effect on production of a *change* in the final demand of 0 for crude and $\$200,000 - \$124,000 = \$76,000$ million for refined products. If we multiply $(I - A)^{-1}$ by the matrix representing this *change*, we obtain

$$\begin{bmatrix} 1.458 & 0.6880 \\ 0.01409 & 1.130 \end{bmatrix} \begin{bmatrix} 0 \\ 76,000 \end{bmatrix} \approx \begin{bmatrix} 53,000 \\ 90,000 \end{bmatrix}$$

$(I - A)^{-1} \times \text{Change in Demand} = \text{Change in Production}$

We see the changes required in production: an increase of \$53,000 million in the crude sector and an increase of \$90,000 million in the refining sector.

Notice that the increase in external demand for the products of the refining sector requires the crude sector to also increase production, even though there is no increase in the *external* demand for its products. The reason is that, in order to increase production, the refining sector needs to use more crude oil, so that the *internal* demand for crude oil goes up. The inverse matrix $(I - A)^{-1}$ takes these **indirect effects** into account in a nice way.

By replacing the \$76,000 by \$1 in the computation we just did, we see that a \$1 increase in external demand for refined products will require an increase in production of \$0.6880 in the crude sector, as well as an increase in production of \$1.130 in the refining sector. This is how we interpret the entries in $(I - A)^{-1}$, and this is why it is useful to look at this matrix inverse rather than just solve $(I - A)X = D$ for X using, say, Gauss-Jordan reduction. Looking at $(I - A)^{-1}$, we can also find the effects of an increase of \$1 in external demand for crude: an increase in production of \$1.458 in the crude sector and an increase of \$0.01409 in the refining sector.

Here are some questions to think about: Why are the diagonal entries of $(I - A)^{-1}$ (slightly) larger than 1? Why is the entry in the lower left so small compared to the others? ■

Interpreting $(I - A)^{-1}$: Indirect Effects

If A is the technology matrix, then the ij th entry of $(I - A)^{-1}$ is the change in the number of units Sector i must produce in order to meet a one-unit increase in external demand for Sector j products. To meet a rising external demand, the necessary change in production for each sector is given by

$$\text{Change in production} = (I - A)^{-1}D^+$$

where D^+ is the change in external demand.

quick Example

Take Sector 1 to be skateboards, and Sector 2 to be wood, and assume that

$$(I - A)^{-1} = \begin{bmatrix} 1.1 & 0 \\ 0.6 & 1.1 \end{bmatrix}$$

Then

$a_{11} = 1.1$ = number of additional units of skateboards that must be produced to meet a one-unit increase in the demand for skateboards (Why is this number larger than 1?)

$a_{12} = 0$ = number of additional units of skateboards that must be produced to meet a one-unit increase in the demand for wood (why is this number 0?)

$a_{21} = 0.6$ = number of additional units of wood that must be produced to meet a one-unit increase in the demand for skateboards

$a_{22} = 1.1$ = number of additional units of wood that must be produced to meet a one-unit increase in the demand for wood

To meet an increase in external demand of 100 skateboards and 400 units of wood, the necessary change in production is

$$(I - A)^{-1}D^+ = \begin{bmatrix} 1.1 & 0 \\ 0.6 & 1.1 \end{bmatrix} \begin{bmatrix} 100 \\ 400 \end{bmatrix} = \begin{bmatrix} 110 \\ 500 \end{bmatrix}$$

so 110 additional skateboards and 500 additional units of wood will need to be produced.

In the preceding examples, we used only two sectors of the economy. The data used in Examples 1 and 2 were taken from an input-output table published by the U.S. Department of Commerce, in which the whole U.S. economy was broken down into 85 sectors. This in turn was a simplified version of a model in which the economy was broken into about 500 sectors. Obviously, computers are required to make a realistic input-output analysis possible. Many governments collect and publish input-output data as part of their national planning. The United Nations collects these data and publishes collections of national statistics. The U.N. also has a useful set of links to government statistics at the following URL:

www.un.org/Depts/unsd/sd_natstat.htm

Example 3 Kenya Economy

Consider four sectors of the economy of Kenya:* (1) the traditional economy, (2) agriculture, (3) manufacture of metal products and machinery, and (4) wholesale and retail trade. The input-output table for these four sectors for 1976 looks like this (all numbers are 1000s of K£):

To		1	2	3	4
From	1	8600	0	0	0
	2	0	20,000	24	0
	3	1500	530	15,000	660
	4	810	8500	5800	2900
	Total Output	87,000	530,000	110,000	180,000



using Technology

See the Technology Guides at the end of the chapter for more on how to do the computations in this example using a TI-83/84 or Excel. Alternatively, go to the Matrix Algebra Tool at

Chapter 3

→ Tools

→ Matrix Algebra Tool

There, type the entries of A as the quotients (column entry/column total—you need not calculate them first), enter D , and then use the format $(I - A)^{-1} * D$.

Suppose that external demand for agriculture increased by K£50,000,000 and that external demand for metal products and machinery increased by K£10,000,000. How would production in these four sectors have to change to meet this rising demand?

Solution To find the change in production necessary to meet the rising demand, we need to use the formula

$$\text{Change in production} = (I - A)^{-1} D^+$$

where A is the technology matrix and D^+ is the change in demand:

$$D^+ = \begin{bmatrix} 0 \\ 50,000 \\ 10,000 \\ 0 \end{bmatrix}$$

With entries shown rounded to two significant digits, the matrix A is

$$A = \begin{bmatrix} 0.099 & 0 & 0 & 0 \\ 0 & 0.038 & 0.00022 & 0 \\ 0.017 & 0.001 & 0.14 & 0.0037 \\ 0.0093 & 0.016 & 0.053 & 0.016 \end{bmatrix} \quad \begin{array}{l} \text{Entries shown rounded to} \\ \text{2 significant digits} \end{array}$$

* Figures are rounded. SOURCE: *Input-Output Tables for Kenya 1976*, Central Bureau of Statistics of the Ministry of Economic Planning and Community Affairs, Kenya.

The next calculation is best done using technology:

$$\text{Change in production} = (I - A)^{-1}D^+ = \begin{bmatrix} 0 \\ 52,000 \\ 12,000 \\ 1500 \end{bmatrix}$$

Entries shown rounded to 2 significant digits

Looking at this result, we see that the changes in external demand will leave the traditional economy unaffected, production in agriculture will rise by K£52 million, production in the manufacture of metal products and machinery will rise by K£12 million, and activity in wholesale and retail trade will rise by K£1.5 million.

† *Before we go on...* Can you see why the traditional economy was unaffected in Example 3? Although it takes inputs from other parts of the economy, it is not itself an input to any other part. In other words, there is no intermediate demand for the products of the traditional economy coming from any other part of the economy, and so an increase in production in any other sector of the economy will require no increase from the traditional economy. On the other hand, the wholesale and retail trade sector does provide input to the agriculture and manufacturing sectors, so increases in those sectors do require an increase in the trade sector.

One more point: Notice how small the off-diagonal entries in $(I - A)^{-1}$ are. This says that increases in each sector have relatively small effects on the other sectors. We say that these sectors are **loosely coupled**. Regional economies, where many products are destined to be shipped out to the rest of the country, tend to show this phenomenon even more strongly. Notice in Example 2 that those two sectors are **strongly coupled**, because a rise in demand for refined products requires a comparable rise in the production of crude. ■

3.5 EXERCISES

● denotes basic skills exercises

tech Ex indicates exercises that should be solved using technology

1. ● Let A be the technology matrix $A = \begin{bmatrix} 0.2 & 0.05 \\ 0.8 & 0.01 \end{bmatrix}$, where

Sector 1 is paper and Sector 2 is wood. Fill in the missing quantities.

- ___ units of wood are needed to produce one unit of paper.
- ___ units of paper are used in the production of one unit of paper.
- The production of each unit of wood requires the use of ___ units of paper.

2. ● Let A be the technology matrix $A = \begin{bmatrix} 0.01 & 0.001 \\ 0.2 & 0.004 \end{bmatrix}$,

where Sector 1 is computer chips and Sector 2 is silicon. Fill in the missing quantities.

- ___ units of silicon are required in the production of one unit of silicon.
 - ___ units of computer chips are used in the production of one unit of silicon.
 - The production of each unit of computer chips requires the use of ___ units of silicon.
3. ● Each unit of television news requires 0.2 units of television news and 0.5 units of radio news. Each unit of radio news requires 0.1 units of television news and no radio news. With Sector 1 as television news and Sector 2 as radio news, set up the technology matrix A .
4. ● Production of one unit of cologne requires no cologne and 0.5 units of perfume. Into one unit of perfume go 0.1 units of cologne and 0.3 units of perfume. With Sector 1 as cologne and Sector 2 as perfume, set up the technology matrix A .

● basic skills

tech Ex technology exercise

In each of Exercises 5–12, you are given a technology matrix A and an external demand vector D . Find the corresponding production vector X . *hint* [see Quick Example on p. 228]

5. ● $A = \begin{bmatrix} 0.5 & 0.4 \\ 0 & 0.5 \end{bmatrix}$, $D = \begin{bmatrix} 10,000 \\ 20,000 \end{bmatrix}$

6. ● $A = \begin{bmatrix} 0.5 & 0.4 \\ 0 & 0.5 \end{bmatrix}$, $D = \begin{bmatrix} 20,000 \\ 10,000 \end{bmatrix}$

7. ● $A = \begin{bmatrix} 0.1 & 0.4 \\ 0.2 & 0.5 \end{bmatrix}$, $D = \begin{bmatrix} 25,000 \\ 15,000 \end{bmatrix}$

8. ● $A = \begin{bmatrix} 0.1 & 0.2 \\ 0.4 & 0.5 \end{bmatrix}$, $D = \begin{bmatrix} 24,000 \\ 14,000 \end{bmatrix}$

9. ● $A = \begin{bmatrix} 0.5 & 0.1 & 0 \\ 0 & 0.5 & 0.1 \\ 0 & 0 & 0.5 \end{bmatrix}$, $D = \begin{bmatrix} 1000 \\ 1000 \\ 2000 \end{bmatrix}$

10. ● $A = \begin{bmatrix} 0.5 & 0.1 & 0 \\ 0.1 & 0.5 & 0.1 \\ 0 & 0 & 0.5 \end{bmatrix}$, $D = \begin{bmatrix} 3000 \\ 3800 \\ 2000 \end{bmatrix}$,

11. ● $A = \begin{bmatrix} 0.2 & 0.2 & 0 \\ 0.2 & 0.4 & 0.2 \\ 0 & 0.2 & 0.2 \end{bmatrix}$, $D = \begin{bmatrix} 16,000 \\ 8000 \\ 8000 \end{bmatrix}$

12. ● $A = \begin{bmatrix} 0.2 & 0.2 & 0.2 \\ 0.2 & 0.4 & 0.2 \\ 0.2 & 0.2 & 0.2 \end{bmatrix}$, $D = \begin{bmatrix} 7000 \\ 14,000 \\ 7000 \end{bmatrix}$

13. ● Given $A = \begin{bmatrix} 0.1 & 0.4 \\ 0.2 & 0.5 \end{bmatrix}$, find the changes in production required to meet an increase in demand of 50 units of Sector 1 products and 30 units of Sector 2 products.

14. ● Given $A = \begin{bmatrix} 0.5 & 0.4 \\ 0 & 0.5 \end{bmatrix}$, find the changes in production required to meet an increase in demand of 20 units of Sector 1 products and 10 units of Sector 2 products.

15. ● Let $(I - A)^{-1} = \begin{bmatrix} 1.5 & 0.1 & 0 \\ 0.2 & 1.2 & 0.1 \\ 0.1 & 0.7 & 1.6 \end{bmatrix}$ and assume that the external demand for the products in Sector 1 increases by 1 unit. By how many units should each sector increase production? What do the columns of the matrix $(I - A)^{-1}$ tell you? *hint* [see Quick Example on p. 232]

16. ● Let $(I - A)^{-1} = \begin{bmatrix} 1.5 & 0.1 & 0 \\ 0.1 & 1.1 & 0.1 \\ 0 & 0 & 1.3 \end{bmatrix}$, and assume that the external demand for the products in each of the sectors increases by 1 unit. By how many units should each sector increase production?

In Exercises 17 and 18, obtain the technology matrix from the given input-output table. *hint* [see Example 1]

17. ●

To	A	B	C
From A	1000	2000	3000
B	0	4000	0
C	0	1000	3000
Total Output	5000	5000	6000

18. ●

To	A	B	C
From A	0	100	300
B	500	400	300
C	0	0	600
Total Output	1000	2000	3000

Applications

19. ● **Campus Food** The two campus cafeterias, the Main Dining Room and Bits & Bytes, typically use each other's food in doing business on campus. One weekend, the input-output table was as follows.³²

To	Main DR	Bits & Bytes
From Main DR	\$10,000	\$20,000
Bits & Bytes	5000	0
Total Output	50,000	40,000

Given that the demand for food on campus last weekend was \$45,000 from the Main Dining Room and \$30,000 from Bits & Bytes, how much did the two cafeterias have to produce to meet the demand last weekend? *hint* [see Example 1]

20. ● **Plagiarism** Two student groups at Enormous State University, the Choral Society and the Football Club, maintain files of term papers that they write and offer to students for research purposes. Some of these papers they use themselves in generating more papers. In order to avoid suspicion of plagiarism by faculty members (who seem to have astute memories), each paper is given to students or used by the clubs only once (no copies are kept). The number of papers that were used in the production of new papers last year is shown in the following input-output table:

To	Choral Soc.	Football Club
From Choral Soc.	20	10
Football Club	10	30
Total Output	100	200

³² For some reason, the Main Dining Room consumes a lot of its own food!

Given that 270 Choral Society papers and 810 Football Club papers will be used by students outside of these two clubs next year, how many new papers do the two clubs need to write?

21. **tech** Ex **Communication Equipment**³³ Two sectors of the U.S. economy are (1) audio, video, and communication equipment and (2) electronic components and accessories. In 1998, the input-output table involving these two sectors was as follows (all figures are in millions of dollars):

To	Equipment	Components
From Equipment	6000	500
Components	24,000	30,000
Total Output	90,000	140,000

Determine the production levels necessary in these two sectors to meet an external demand for \$80,000 million of communication equipment and \$90,000 million of electronic components. Round answers to two significant digits.

22. **tech** Ex **Wood and Paper**³⁴ Two sectors of the U.S. economy are (1) lumber and wood products and (2) paper and allied products. In 1998 the input-output table involving these two sectors was as follows (all figures are in millions of dollars).

To	Wood	Paper
From Wood	36,000	7000
Paper	100	17,000
Total Output	120,000	120,000

If external demand for lumber and wood products rises by \$10,000 million and external demand for paper and allied products rises by \$20,000 million, what increase in output of these two sectors is necessary? Round answers to two significant digits.

23. ● **Australia Economy**³⁵ Two sectors of the Australian economy are (1) textiles and (2) clothing and footwear. The 1977 input-output table involving these two sectors results in the following value for $(I - A)^{-1}$:

$$(I - A)^{-1} = \begin{bmatrix} 1.228 & 0.182 \\ 0.006 & 1.1676 \end{bmatrix}$$

Complete the following sentences.

- a. ____ additional dollars worth of clothing and footwear must be produced to meet a \$1 increase in the demand for textiles.
- b. 0.182 additional dollars worth of ____ must be produced to meet a one-dollar increase in the demand for ____.

³³ The data have been rounded. SOURCE: *Survey of Current Business*, December, 2001, U.S. Department of Commerce.

³⁴ Ibid.

³⁵ SOURCE: *Australian National Accounts and Input-Output Tables 1978-1979*, Australian Bureau of Statistics.

24. **Australia Economy**³⁶ Two sectors of the Australian economy are (1) community services and (2) recreation services. The 1978-79 input-output table involving these two sectors results in the following value for $(I - A)^{-1}$:

$$(I - A)^{-1} = \begin{bmatrix} 1.0066 & 0.00576 \\ 0.00496 & 1.04206 \end{bmatrix}$$

Complete the following sentences.

- a. 0.00496 additional dollars worth of ____ must be produced to meet a \$1 increase in the demand for ____.
- b. ____ additional dollars worth of community services must be produced to meet a one-dollar increase in the demand for community services.

tech Ex Exercises 25-28 require the use of technology.

25. **tech** Ex **United States Input-Output Table**³⁷ Four sectors of the U.S. economy are (1) livestock and livestock products, (2) other agricultural products, (3) forestry and fishery products, and (4) agricultural, forestry, and fishery services. In 1977 the input-output table involving these four sectors was as follows (all figures are in millions of dollars):

To	1	2	3	4
From 1	11,937	9	109	855
2	26,649	4285	0	4744
3	0	0	439	61
4	5423	10,952	3002	216
Total Output	97,795	120,594	14,642	47,473

Determine how these four sectors would react to an increase in demand for livestock (Sector 1) of \$1000 million, how they would react to an increase in demand for other agricultural products (Sector 2) of \$1000 million, and so on.

26. **tech** Ex **United States Input-Output Table**³⁸ Four sectors of the U.S. economy are (1) motor vehicles, (2) truck and bus bodies, trailers, and motor vehicle parts, (3) aircraft and parts, and (4) other transportation equipment. In 1998 the input-output table involving these four sectors was (all figures in millions of dollars):

To	1	2	3	4
From 1	75	1092	0	1207
2	64,858	13,081	7	1070
3	0	0	21,782	0
4	0	0	0	1375
Total Output	230,676	135,108	129,376	44,133

³⁶ Ibid.

³⁷ SOURCE: *Survey of Current Business* December, 2001, U.S. Department of Commerce.

³⁸ Ibid.

Determine how these four sectors would react to an increase in demand for motor vehicles (Sector 1) of \$1000 million, how they would react to an increase in demand for truck and bus bodies (Sector 2) of \$1000 million, and so on.

- 27. Australia Input-Output Table** Four sectors of the Australian economy are (1) agriculture, (2) forestry, fishing, and hunting, (3) meat and milk products, and (4) other food products. In 1978–79 the input-output table involving these four sectors was as follows (all figures are in millions of Australian dollars).³⁹

To		1	2	3	4
From	1	678.4	3.7	3341.5	1023.5
	2	15.5	6.9	17.1	124.5
	3	47.3	4.3	893.1	145.8
	4	312.5	22.1	83.2	693.5
	Total Output	9401.3	685.8	6997.3	4818.3

- a. How much additional production by the meat and milk sector is necessary to accommodate a \$100 increase in the demand for agriculture?
 b. Which sector requires the most of its own product in order to meet a \$1 increase in external demand for that product?

- 28. Australia Input-Output Table** Four sectors of the Australian economy are (1) petroleum and coal products, (2) nonmetallic mineral products, (3) basic metals and products, and (4) fabricated metal products. In 1978–79 the input-output table involving these four sectors was as follows (all figures are in millions of Australian dollars).⁴⁰

To		1	2	3	4
From	1	174.1	30.5	120.3	14.2
	2	0	190.1	55.8	12.6
	3	2.1	40.2	1418.7	1242.0
	4	0.1	7.3	40.4	326.0
	Total Output	3278.0	2188.8	6541.7	4065.8

- a. How much additional production by the petroleum and coal products sector is necessary to accommodate a \$1000 increase in the demand for fabricated metal products?
 b. Which sector requires the most of the product of some other sector in order to meet a \$1 increase in external demand for that product?

Communication and Reasoning Exercises

29. ● What would it mean if the technology matrix A were the zero matrix?
 30. ● Can an external demand be met by an economy whose technology matrix A is the identity matrix? Explain.
 31. What would it mean if the total output figure for a particular sector of an input-output table were equal to the sum of the figures in the row for that sector?
 32. What would it mean if the total output figure for a particular sector of an input-output table were less than the sum of the figures in the row for that sector?
 33. What does it mean if an entry in the matrix $(I - A)^{-1}$ is zero?
 34. Why do we expect the diagonal entries in the matrix $(I - A)^{-1}$ to be slightly larger than 1?
 35. Why do we expect the off-diagonal entries of $(I - A)^{-1}$ to be less than 1?
 36. Why do we expect all the entries of $(I - A)^{-1}$ to be non-negative?

³⁹SOURCE: *Australian National Accounts and Input-Output Tables 1978–1979*, Australian Bureau of Statistics.

⁴⁰ Ibid.

Chapter 3 Review

KEY CONCEPTS

3.1 Matrix Addition and Scalar Multiplication

$m \times n$ matrix, dimensions, entries. *p.* 174
Referring to the entries of a matrix *p.* 174

Matrix equality *p.* 175

Row, column, and square matrices *p.* 175

Addition and subtraction of matrices *p.* 176

Scalar multiplication *p.* 177

Properties of matrix addition and scalar multiplication *p.* 179

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3.2 Matrix Multiplication

Multiplying a row by a column *p.* 184

Linear equation as a matrix equation *p.* 186

The product of two matrices: general case *p.* 186

Identity matrix *p.* 190

Properties of matrix addition and multiplication *p.* 191

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A system of linear equations can be written as a single matrix equation *p.* 192

3.3 Matrix Inversion

The inverse of a matrix, singular matrix *p.* 199

Procedure for finding the inverse of a matrix *p.* 200

Formula for the inverse of a 2×2 matrix; determinant of a 2×2 matrix *p.* 201

Using an inverse matrix to solve a system of equations. *p.* 203

3.4 Game Theory

Two-person zero sum game, payoff matrix *p.* 210

A strategy specifies how a player chooses a move *p.* 210

The expected value of a game for given mixed strategies R and C *p.* 210

An optimal strategy, according to the minimax criterion, is one that mini-

mizes the maximum damage your opponent can cause you. *p.* 214

The Fundamental Principle of Game Theory *p.* 214

Procedure for reducing by dominance *p.* 215

Procedure for solving a 2×2 game *p.* 215

The expected value of a game is its expected value when the players use their optimal strategies. *p.* 218

A strictly determined game is one with a saddle point. *p.* 219

Steps to follow in solving a game *p.* 220

3.5 Input-Output Models

An input-output model divides an economy into sectors. The technology matrix records the interactions of these sectors and allows us to relate external demand to the production vector. *p.* 228

Procedure for finding a technology matrix from an input-output table *p.* 230

The entries of $(I - A)^{-1}$ *p.* 232

REVIEW EXERCISES

For Exercises 1–10, let

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, B = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix},$$

$$C = \begin{bmatrix} -1 & 0 \\ 1 & 1 \\ 0 & 1 \end{bmatrix} \text{ and } D = \begin{bmatrix} -3 & -2 & -1 \\ 1 & 2 & 3 \end{bmatrix}$$

For each of the following, determine whether the expression is defined, and if it is, evaluate it.

1. $A + B$

2. $A - D$

3. $2A^T + C$

4. AB

5. $A^T B$

6. A^2

7. B^2

8. B^3

9. $AC + B$

10. $CD + B$

For each matrix in Exercises 11–16, find the inverse or determine that the matrix is singular.

11. $\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

12. $\begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$

13. $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

14. $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

15. $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 3 & 3 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix}$

16. $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

Write each system of linear equations in Exercises 17–20 as a matrix equation, and solve by inverting the coefficient matrix.

17. $x + 2y = 0$
 $3x + 4y = 2$

18. $x + y + z = 3$
 $y + 2z = 4$
 $y - z = 1$

19. $x + y + z = 2$
 $x + 2y + z = 3$
 $x + y + 2z = 1$

20. $x + y = 0$
 $y + z = 1$
 $z + w = 0$
 $x - w = 3$

In each of Exercises 21–24, solve the game with the given matrix and give the expected value of the game.

$$21. P = \begin{bmatrix} 2 & 1 & 3 & 2 \\ -1 & 0 & -2 & 1 \\ 2 & 0 & 1 & 3 \end{bmatrix} \quad 22. P = \begin{bmatrix} 3 & -3 & -2 \\ -1 & 3 & 0 \\ 2 & 2 & 1 \end{bmatrix}$$

$$23. P = \begin{bmatrix} -1 & -3 & -2 \\ -1 & 3 & 0 \\ 3 & 3 & -1 \end{bmatrix} \quad 24. P = \begin{bmatrix} 1 & 4 & 3 & 3 \\ 0 & -1 & 2 & 3 \\ 2 & 0 & -1 & 2 \end{bmatrix}$$

In each of Exercises 25–28, find the production vector X corresponding to the given technology matrix A and external demand vector D .

$$25. A = \begin{bmatrix} 0.3 & 0.1 \\ 0 & 0.3 \end{bmatrix}, D = \begin{bmatrix} 700 \\ 490 \end{bmatrix}$$

$$26. A = \begin{bmatrix} 0.7 & 0.1 \\ 0.1 & 0.7 \end{bmatrix}, D = \begin{bmatrix} 1000 \\ 2000 \end{bmatrix}$$

$$27. A = \begin{bmatrix} 0.2 & 0.2 & 0.2 \\ 0 & 0.2 & 0.2 \\ 0 & 0 & 0.2 \end{bmatrix}, D = \begin{bmatrix} 32,000 \\ 16,000 \\ 8,000 \end{bmatrix}$$

$$28. A = \begin{bmatrix} 0.5 & 0.1 & 0 \\ 0.1 & 0.5 & 0.1 \\ 0 & 0.1 & 0.5 \end{bmatrix}, D = \begin{bmatrix} 23,000 \\ 46,000 \\ 23,000 \end{bmatrix}$$

Applications

It is now July 1 and online sales of romance, science fiction, and horror novels at OHaganBooks.com were disappointingly slow over the past month. Exercises 29 and 30 are based on the following tables:

Inventory of books in stock on June 1 at the OHaganBooks.com warehouses in Texas and Nevada:

Books in Stock (June 1)

	Romance	Sci Fi	Horror
Texas	2500	4000	3000
Nevada	1500	3000	1000

Online sales during June:

Online Sales (June)

	Romance	Sci Fi	Horror
Texas	300	500	100
Nevada	100	450	200

29. Inventory Use matrix algebra to compute the inventory at each warehouse at the end of June.

30. Inventory It is now July 15. Based on July sales to date, the e-commerce manager projects July sales as follows:

July Sales (Projected)

	Romance	Sci Fi	Horror
Texas	280	550	100
Nevada	50	500	120

Assuming that sales continue at this level for the next few months, write down a matrix equation showing the inventory N at each warehouse x months after July 1 (assuming no new books are ordered during that period).

31. Revenue It is now the end of July and OHaganBooks.com's e-commerce manager bursts into the CEO's office. "I thought you might want to know, John, that our sales figures are exactly what I projected two weeks ago (see Exercise 30). Is that good market analysis or what?" OHaganBooks.com has charged an average of \$5 for romance novels, \$6 for science fiction novels, and \$5.50 for horror novels. Use the projected sales figures from Exercise 30 and matrix arithmetic to compute the total revenue OHaganBooks.com earned at each warehouse in July.

32. Revenue OHaganBooks.com pays an average of \$2 for romance novels, \$3.50 for science fiction novels, and \$1.50 for horror novels. Use this information together with the information in Exercise 31 to compute the profit OHaganBooks.com earned from sales of these books at each warehouse in July.

OHaganBooks.com has two main competitors: JungleBooks.com and FarmerBooks.com, and no other competitors of any significance on the horizon. Exercises 33–35 are based on the following table, which shows the movement of customers during July.⁴¹ (Thus, for instance, the first row tells us that 80 percent of OHaganBooks.com's customers remained loyal, 10 percent of them went to JungleBooks.com, and the remaining 10 percent to FarmerBooks.com.)

	To OHagan	To Jungle	To Farmer
From OHagan	0.8	0.1	0.1
From Jungle	0.4	0.6	0
From Farmer	0.2	0	0.8

At the beginning of July, OHaganBooks.com had an estimated 2000 customers, while its two competitors had 4000 each.

33. Competition Set up the July 1 customer numbers in a row matrix, and use matrix arithmetic to estimate the number of customers each company has at the end of July.

⁴¹ By a "customer" of one of the three e-commerce sites, we mean someone who purchases more at that site than at any of the two competitors.

34. Competition Assuming the July trends continue in August, predict the number of customers each company will have at the end of August.

35. Competition Name one or more important factors that the model we have used in this question does not take into account.

Acting on a “tip” from Marjory Duffin, John O’Hagan decided that his company should invest a significant sum in the stock of publisher Duffin House. Exercises 36 and 37 are based on the following table, which shows what information he pieced together later, after some of the records had been lost.

Date	Number of Shares	Price per Share
July 1	?	\$20
August 1	?	\$10
September 1	?	\$5
Total	5000	

Over the three months shown, the company invested a total of \$50,000 in Duffin stock, and, on August 15, was paid dividends of 10¢ per share, for a total of \$300.

36. Investments Use matrix inversion to determine how many shares OHaganBooks.com purchased on each of the three dates shown.

37. Investments (Refer to Exercise 36.) On October 1, the shares purchased on July 1 were sold at \$3 per share. The remaining shares were sold one month later at \$1 per share. Use matrix algebra to determine the total loss (taking into account the dividends paid on August 15) incurred as a result of the Duffin stock fiasco.

Publisher Marjory Duffin reveals that JungleBooks may be launching a promotional scheme in which it will offer either two books for the price of one, or three books for the price of two (Marjory can’t quite seem to remember which, and is not certain whether they will go with the scheme at all). Your marketing advisers Floody and O’Lara seem to have different ideas as to how to respond. Floody suggests you counter by offering *three* books for the price of one, while O’Lara suggests that you instead offer a free copy of the “Finite Mathematics Student Solutions Manual” with every purchase. After a careful analysis, you come up with the following payoff matrix, where the payoffs represent the number of customers, in thousands, you expect to gain from JungleBooks.

		JungleBooks		
		No Promo	2 for Price of 1	3 for Price of 2
H’Hagan	No Promo	0	−60	−40
	3 for Price of 1	30	20	10
	Finite Math	20	0	15

Use the above information in Exercises 38–40.

38. Competition Determine whether the game is strictly determined. (If so, what should you do?)

39. Competition After a very expensive dinner at an exclusive restaurant, Marjory suddenly “remembers” that the JungleBooks CEO mentioned to her (at a less expensive restaurant) that there is only a 20 percent chance JungleBooks will launch a “2 for the price of 1” promotion, and a 40 percent chance that it will launch a “3 for the price of 2” promotion. What should you do in view of this new information, and what will the expected effect be on your customer base?

40. Competition You are about to go with the option chosen in the preceding exercise when one of your corporate spies reveals that Marjory Duffin has just been seen at the lavish Donald Club 8000 in earnest conversation with JungleBooks CEO François Dubois. All bets are off; you can assume now that JungleBooks knows what options you’re considering and knows the payoff matrix above. *Now* what should you do, and how many customers should you expect to gain or lose?

Some of the books sold by OHaganBooks.com are printed at Bruno Mills, Inc., a combined paper mill and printing company. Exercises 41–44 are based on the following typical monthly input-output table for Bruno Mills’ paper and book printing sectors.

		Paper	Books
From	Paper	\$20,000	\$50,000
	Books	2,000	5,000
Total Output		200,000	100,000

41. Production Find the technology matrix for Bruno Mills’ paper and book printing sectors.

42. Production Compute $(I - A)^{-1}$. What is the significance of the (1, 2) entry?

43. Production Approximately \$1700 worth of the books sold each month by OHaganBooks.com are printed at Bruno Mills, Inc., and OHaganBooks.com uses approximately \$170 worth of Bruno Mills’ paper products each month. What is the total value of paper and books that must be produced by Bruno Mills, Inc. in order to meet demand from OHaganBooks.com?

44. Production Currently, Bruno Mills, Inc. has a monthly capacity of \$500,000 of paper products and \$200,000 of books. What level of external demand would cause Bruno to meet the capacity for both products?



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CASE STUDY: The Japanese Economy



Jose Fuste Rajar/zefa/Corbis

A senator walks into your cubicle in the Congressional Budget Office. “Look here,” she says, “I don’t see why the Japanese trade representative is getting so upset with my proposal to cut down on our use of Japanese finance and insurance. He claims that it’ll hurt Japan’s mining operations. But just look at Japan’s input-output table. The finance sector doesn’t use any input from the mining sector. How can our cutting back on finance and insurance hurt mining?” Indeed, the senator is right about the input-output table, which

you have hanging on your wall. Here is what it looks like (all figures are in 100 million yen).⁴²

	1	2	3	4	5	6	7	8	9	10	11	12	13
1	19,221	8	99,417	1,610	0	100	0	1	23	0	20	12,491	0
2	0	41	53,006	8,187	13,189	0	0	0	0	0	7	45	7
3	25,376	959	1,247,342	259,049	14,488	38,247	13,316	1,635	54,426	3,827	26,614	267,704	4,963
4	503	105	13,909	2,242	11,664	5,924	1,340	22,788	4,712	1,591	4,627	11,793	0
5	716	471	59,111	6,203	25,035	11,654	1,940	2,265	8,764	1,811	8,528	46,068	915
6	6,559	291	171,655	61,848	3,146	11,242	2,225	1,066	18,053	762	4,683	78,454	1,123
7	5,303	733	43,394	9,533	7,238	58,662	35,348	32,706	30,879	2,252	824	53,827	9,003
8	42	157	11,353	2,731	2,533	38,416	6,772	4,790	8,308	2,445	498	27,639	735
9	7,273	4,021	93,244	46,994	6,781	53,416	7,046	1,623	52,905	4,151	8,369	38,765	1,416
10	135	76	8,682	4,913	1,118	19,014	6,740	435	3,465	9,167	3,826	37,155	98
11	0	0	0	0	0	0	0	0	0	0	0	0	4,614
12	1,780	688	207,289	69,945	24,464	53,225	37,826	10,267	65,577	19,902	18,595	142,920	3,327
13	1,507	228	23,154	1,788	1,702	5,874	1,456	5,112	2,286	1,196	4,263	11,511	0

The sectors are

1. Agriculture, forestry, and fishery
2. Mining
3. Manufacturing
4. Construction
5. Electric power, gas & water supply
6. Commerce
7. Finance and insurance
8. Real estate
9. Transport
10. Communication and broadcasting

⁴² SOURCE: 1995 *Input-Output Tables for Japan*, Management and Coordination Agency, Government of Japan, March 2000. Obtained from www.stat.go.jp.

11. Public administration
12. Services
13. Activities not elsewhere classified

The total output from each sector is given in the following table:

1	2	3	4	5	6	7	8	9	10	11	12	13
158,178	16,595	3,145,585	881,493	264,635	1,023,216	363,346	641,852	501,138	147,628	262,170	1,909,996	55,176

“If I look at just the mining and finance sectors,” says the senator, “I’m looking at this input-output table.”

		To	Mining (2)	Finance (7)
From	Mining (2)		41	0
	Finance (7)		733	35,348
Total Output			16,595	363,346

“That gives me

$$A = \begin{bmatrix} 0.0025 & 0 \\ 0.0442 & 0.0973 \end{bmatrix}$$

and so

$$(I - A)^{-1} = \begin{bmatrix} 1.0025 & 0 \\ 0.0491 & 1.1078 \end{bmatrix}$$

That last column tells me that any change in demand for finance will have no effect on demand for mining.”

Now you have to explain to the senator a point that we fudged a bit in Section 3.5. What she said assumes that changing the external demand for finance (that is, the demand from outside of these two sectors) will not change the external demand for mining. But in fact, that is unlikely to be true. Changing the demand for finance will change the demand for other sectors in the economy directly (the manufacturing sector, for example), which in turn may change the demand for mining. To see these indirect effects properly, you tell the senator that she must look at the whole Japanese economy. The technology matrix A is then

0.121515	0.000482	0.031605	0.001826	0	0.000098	0	0.000002	0.000046	0	0.000076	0.006540	0
0	0.002471	0.016851	0.009288	0.049838	0	0	0	0	0	0.000027	0.000024	0.000127
0.160427	0.057788	0.396537	0.293875	0.054747	0.037379	0.036648	0.002547	0.108605	0.025923	0.101514	0.140159	0.089949
0.003180	0.006327	0.004422	0.002543	0.044076	0.005790	0.003688	0.035504	0.009403	0.010777	0.017649	0.006174	0
0.004527	0.028382	0.018792	0.007037	0.094602	0.011390	0.005339	0.003529	0.017488	0.012267	0.032529	0.024119	0.016583
0.041466	0.017535	0.054570	0.070163	0.011888	0.010987	0.006124	0.001661	0.036024	0.005162	0.017862	0.041075	0.020353
0.033526	0.044170	0.013795	0.010815	0.027351	0.057331	0.097285	0.050956	0.061618	0.015255	0.003143	0.028182	0.163169
0.000266	0.009461	0.003609	0.003098	0.009572	0.037544	0.018638	0.007463	0.016578	0.016562	0.001900	0.014471	0.013321
0.045980	0.242302	0.029643	0.053312	0.025624	0.052204	0.019392	0.002529	0.105570	0.028118	0.031922	0.020296	0.025663
0.000853	0.004580	0.002760	0.005573	0.004225	0.018583	0.018550	0.000678	0.006914	0.062095	0.014594	0.019453	0.001776
0	0	0	0	0	0	0	0	0	0	0	0	0.083623
0.011253	0.041458	0.065898	0.079348	0.092444	0.052017	0.104105	0.015996	0.130856	0.134812	0.070927	0.074827	0.060298
0.009527	0.013739	0.007361	0.002028	0.006431	0.005741	0.004007	0.007964	0.004562	0.008101	0.016260	0.006027	0

and $(I - A)^{-1}$ is

1.151615	0.008820	0.064373	0.023826	0.008161	0.004945	0.005439	0.001751	0.011777	0.005486	0.009378	0.019111	0.009187
0.006998	1.008375	0.032070	0.020476	0.059556	0.003058	0.002844	0.001357	0.006714	0.003262	0.006507	0.007120	0.005696
0.354517	0.202960	1.754282	0.569941	0.190926	0.114240	0.119857	0.039317	0.281756	0.113835	0.232270	0.296456	0.228125
0.008384	0.014417	0.013625	1.010041	0.053425	0.010797	0.007977	0.037093	0.016950	0.015866	0.023000	0.012313	0.007258
0.018575	0.046643	0.046090	0.028805	1.117416	0.020504	0.014828	0.006923	0.035840	0.023719	0.046783	0.039408	0.032861
0.075374	0.047784	0.112144	0.115655	0.038606	1.026034	0.022400	0.009075	0.068981	0.023330	0.041195	0.067975	0.044748
0.065086	0.085917	0.054440	0.045494	0.055681	0.080239	1.122213	0.062448	0.098977	0.034915	0.023922	0.053391	0.199228
0.008625	0.021395	0.016655	0.015301	0.018399	0.044074	0.025858	1.010222	0.028489	0.023738	0.008612	0.022987	0.023259
0.081959	0.291054	0.084629	0.098878	0.065802	0.071023	0.036062	0.010081	1.143621	0.046935	0.055252	0.046613	0.053068
0.007101	0.013239	0.012988	0.015212	0.011790	0.024942	0.026509	0.003336	0.017217	1.072079	0.020839	0.027392	0.011983
0.001291	0.001568	0.001402	0.000791	0.000964	0.000731	0.000602	0.000760	0.000851	0.000969	1.001677	0.000902	0.084107
0.067590	0.122810	0.160730	0.160074	0.153207	0.093094	0.148358	0.034321	0.206047	0.181389	0.117705	1.129124	0.127110
0.015436	0.018751	0.016768	0.009458	0.011522	0.008746	0.007199	0.009085	0.010175	0.011591	0.020055	0.010782	1.005784

You tell the senator to look at the 7th column to see the effects of a change in demand for finance. There is indeed an effect on mining: Every 1 yen increase in demand for finance produces a 0.002844 yen increase in demand for mining. This also means that every 1 yen *decrease* in demand for finance produces a 0.002844 yen *decrease* in demand for mining. So the Japanese trade representative is right to complain that the senator's plan will hurt their mining companies.

Exercises

1. What does the (2, 7) entry in the matrix A tell you?
2. What does the (2, 7) entry in the matrix $(I - A)^{-1}$ tell you?
3. Why are none of the entries of $(I - A)^{-1}$ negative?
4. Why are the diagonal entries of $(I - A)^{-1}$ close to, and larger than, 1?
5. An increase in demand for the products of which sector of the Japanese economy would have the least impact on the mining sector?
6. An increase in demand for the products of which sector of the Japanese economy would have the most impact on the mining sector?
7. Which sector of the Japanese economy has the greatest percentage of its total output available for external consumption?
8. Referring to the conclusion of this section, try to account for most of the 0.002844 yen by looking at the technology matrix A . For example, a 1 yen increase in demand for finance produces directly a 0.036648 yen increase in demand for manufacturing, which in turn produces a $(0.036648)(0.016851) = 0.000618$ yen increase in demand for mining. What other two-step effects are there? Do they account for all of the 0.002844 yen?
9. Explain why A^2 gives you the total two-step effects of increases in each sector on the others. What about three-step effects? How do you see *all* of the direct and indirect effects?

TI-83/84 Technology Guide

On the TI-83/84, matrices are referred to as $[A]$, $[B]$, and so on through $[J]$. To enter a matrix, press **MATRIX** to bring up the matrix menu, select EDIT, select a matrix, and press **ENTER**. Then enter the dimensions of the matrix followed by its entries. When you want to use a matrix, press **MATRIX**, select the matrix and press **ENTER**.

Section 3.1

Example 2 The A-Plus auto parts store chain has two outlets, one in Vancouver and one in Quebec. Among other things, it sells wiper blades, windshield cleaning fluid, and floor mats. The monthly sales of these items at the two stores for two months are given in the following tables:

January Sales

	Vancouver	Quebec
Wiper Blades	20	15
Cleaning Fluid (bottles)	10	12
Floor Mats	8	4

February Sales

	Vancouver	Quebec
Wiper Blades	23	12
Cleaning Fluid (bottles)	8	12
Floor Mats	4	5

Use matrix arithmetic to calculate the change in sales of each product in each store from January to February.

Solution with Technology On the TI-83/84, adding matrices is similar to adding numbers. The sum of the matrices $[A]$ and $[B]$ is $[A] + [B]$; their difference, of course, is $[A] - [B]$. As in the text, for this example,

1. Create two matrices, $[J]$ and $[F]$.
2. Compute their difference, $[F] - [J]$ using $[F] - [J] \rightarrow [D]$.

MATRIX[J] 3 × 2
 $\begin{bmatrix} 20 & 15 \\ 10 & 12 \\ 8 & 4 \end{bmatrix}$
 3, 2=4

MATRIX[F] 3 × 2
 $\begin{bmatrix} 23 & 12 \\ 8 & 12 \\ 4 & 5 \end{bmatrix}$
 3, 2=5

$[F] - [J] \rightarrow [D]$
 $\begin{bmatrix} 3 & -3 \\ -2 & 0 \\ -4 & 1 \end{bmatrix}$

Note that we have stored the difference in the matrix $[D]$ in case we need it for later use.

Example 3 The revenue generated by sales in the Vancouver and Quebec branches of the A-Plus auto parts store was as follows:

January Sales in Canadian Dollars

	Vancouver	Quebec
Wiper Blades	140.00	105.00
Cleaning Fluid (bottles)	30.00	36.00
Floor Mats	96.00	48.00

If the Canadian dollar was worth \$0.65US at the time, compute the revenue in U.S. dollars.

Solution with Technology Scalar multiplication on the TI-83/84 is similar to multiplication of numbers.

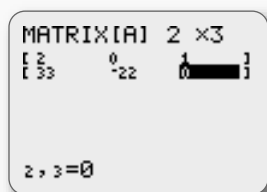
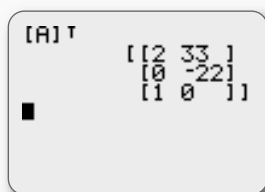
1. Enter the matrix A using the matrix editor.

MATRIX[A] 3 × 2
 $\begin{bmatrix} 140 & 105 \\ 30 & 36 \\ 96 & 48 \end{bmatrix}$
 3, 2=48

2. Calculate $0.65A$ using the formula $0.65 * [A]$ or just $0.65 [A]$.

$0.65 * [A]$
 $\begin{bmatrix} 91 & 68.25 \\ 19.5 & 23.4 \\ 62.4 & 31.2 \end{bmatrix}$

Transposition On the TI-83/84 you can access the symbol “T” by pressing $\boxed{\text{MATRX}}$, selecting MATH, selecting T, and then pressing $\boxed{\text{ENTER}}$. Here is an example:

The matrix A Computation of $[A]^T$

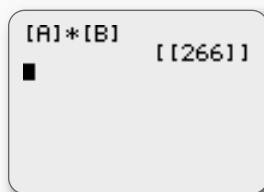
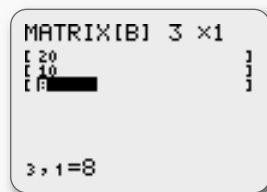
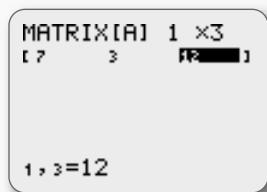
Section 3.2

Example 1 The A-Plus auto parts store had the following sales in its Vancouver store.

	Vancouver
Wiper Blades	20
Cleaning Fluid (bottles)	10
Floor Mats	8

The store sells wiper blades for \$7.00 each, cleaning fluid for \$3.00 per bottle, and floor mats for \$12.00 each. Use matrix multiplication to find the total revenue generated by sales of these items.

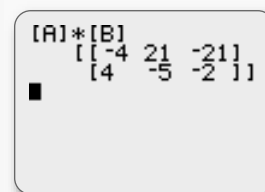
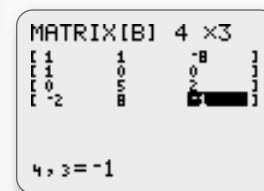
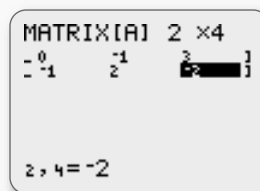
Solution with Technology On the TI-83/84, the format for multiplying matrices is the same as for multiplying numbers: $[A][B]$ or $[A] * [B]$ will give the product. Because we can't use P and Q as names of matrices on the TI-83/84, we'll use $[A]$ for P and $[B]$ for Q . We enter the matrices and then multiply them:



Example 3 Calculate

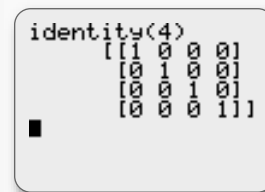
$$\begin{bmatrix} 2 & 0 & -1 & 3 \\ 1 & -1 & 2 & -2 \end{bmatrix} \begin{bmatrix} 1 & 1 & -8 \\ 1 & 0 & 0 \\ 0 & 5 & 2 \\ -2 & 8 & -1 \end{bmatrix}$$

Solution with Technology We can do this computation just as we did the one in Example 1. Note that if you try to multiply two matrices whose product is not defined, you will get the error “DIM MISMATCH” (dimension mismatch).



Note that, while editing, you can see only three columns of $[A]$ at a time.

Example 6 —Identity Matrix On the TI-83/84, the function $\text{identity}(n)$ (in the $\boxed{\text{MATRX}}$ MATH menu) returns the $n \times n$ identity matrix.

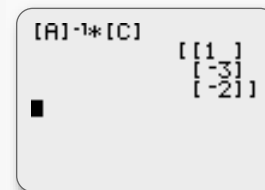
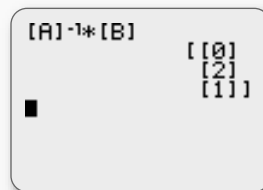
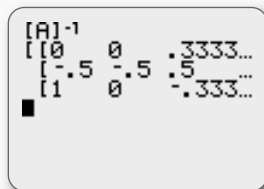
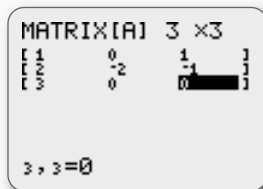


Section 3.3

Example 2 Find the inverse of

$$Q = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -2 & -1 \\ 3 & 0 & 0 \end{bmatrix}$$

Solution with Technology On a TI-83/84, you can invert the square matrix $[A]$ by entering $[A] [x^{-1}]$ ENTER .



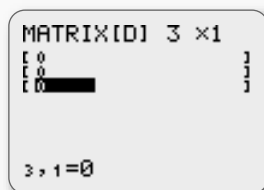
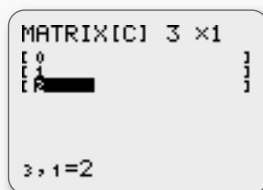
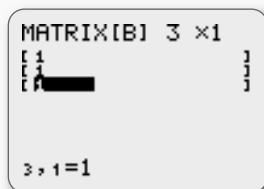
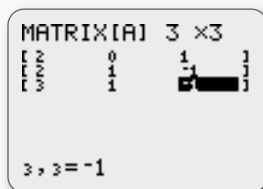
(Note that you can use the right and left arrow keys to scroll the answer, which cannot be shown on the screen all at once.) You could also use the calculator to help you go through the row reduction, as described in Chapter 2.

Example 4 Solve the following three systems of equations.

- a. $2x + z = 1$
 $2x + y - z = 1$
 $3x + y - z = 1$
- b. $2x + z = 0$
 $2x + y - z = 1$
 $3x + y - z = 2$
- c. $2x + z = 0$
 $2x + y - z = 0$
 $3x + y - z = 0$

Solution with Technology

1. Enter the four matrices A , B , C , and D :



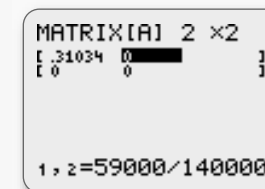
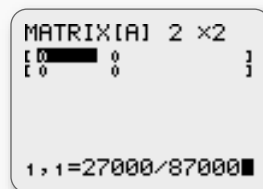
Section 3.5

Example 1 Recall that the input-output table in Example 1 looks like this:

		To	Crude	Refining
From	Crude		27,000	59,000
	Refining		750	15,000
Total Output			87,000	140,000

What was the technology matrix for these two sectors? What was left over from each of these sectors for use by other parts of the economy or for export?

Solution with Technology There are several ways to use these data to create the technology matrix in your TI-83/84. For small matrices like this, the most straightforward is to use the matrix editor, where you can give each entry as the appropriate quotient:



and so on. Once we have the technology matrix [A] and the production vector [B] (remember that we can't use [X] as a matrix name), we can calculate the external demand vector:

```
[B]-[A]*[B]
[[1000 ]
 [124250]]
```

Of course, when interpreting these numbers, we must remember to round to two significant digits, because our original data were accurate to only that many digits.

Here is an alternative way to calculate the technology matrix that may be better for examples with more sectors.

1. Begin by entering the columns of the input-output table as lists in the list editor (**STAT** EDIT).

```
L1 | L2 | L3 | 2
---|---|---|---
27000 | 58000 | ----- |
750 | 140000 | ----- |
---|---|---|---
L2(2) = 15000
```

2. We now want to divide each column by the total output of its sector and assemble the results into a matrix. We can do this using the **List** ► **matr** function (under the **MATRIX** MATH menu) as follows:

```
List►matr(L1/870
00,L2/140000,[A]
)
Done
[A]
[[.3103448276 ...
 [.0086206897 ...
```

Example 3 Consider four sectors of the economy of Kenya:⁴³ (1) the traditional economy, (2) agriculture, (3) manufacture of metal products and machinery, and (4) wholesale and retail trade. The input-output table for these

four sectors for 1976 looks like this (all numbers are 1000s of K£):

To	1	2	3	4
From 1	8600	0	0	0
2	0	20,000	24	0
3	1500	530	15,000	660
4	810	8500	5800	2900
Total Output	87,000	530,000	110,000	180,000

Suppose that external demand for agriculture increased by K£50,000,000 and that external demand for metal products and machinery increased by K£10,000,000. How would production in these four sectors have to change to meet this rising demand?

Solution with Technology

1. Enter the technology matrix [A] using one of the techniques above.

```
MATRIX[A] 4 × 4
[ 0 0 0 0 ]
[-.03774 2.2E-4 0 0 ]
[-.001 .13636 .00367 ]
[-.01604 .05273 .00454 ]
4 × 4 = .01611111111...
```

2. Enter the matrix D^+ as [D].

```
MATRIX[D] 4 × 1
[ 0 ]
[ 50000 ]
[ 10000 ]
[ 0 ]
4 × 1 = 0
```

3. You can then compute the change in production with the following formula:

$$(\text{identity}(4) - [A])^{-1} [D]$$

```
(<identity(4)-[A]
>^-1[D]
[[0 ]
 [51963.42476]
 [11645.36129]
 [1471.104956]]
```

⁴³ Figures are rounded. SOURCE: *Input-Output Tables for Kenya 1976*, Central Bureau of Statistics of the Ministry of Economic Planning and Community Affairs, Kenya.

EXCEL Technology Guide

Section 3.1

To enter a matrix in a spreadsheet, we put its entries in any convenient block of cells. For example, the matrix A in the first Quick Example of this section might look like this:

	A	B	C	D
1	2	0	1	
2	33	-22	0	
3				
4				

Excel refers to such blocks of data as **arrays**, which it can handle in much the same way as it handles single cells of data. For instance, when typing a formula, just as clicking on a cell creates a reference to that cell, selecting a whole array of cells will create a reference to that array. An array is referred to using an **array range** consisting of the top-left and bottom-right cell coordinates, separated by a colon. For example, the array range A1:C2 refers to the 2×3 matrix above, with top-left corner A1 and bottom-right corner C2.

Example 2 The A-Plus auto parts store chain has two outlets, one in Vancouver and one in Quebec. Among other things, it sells wiper blades, windshield cleaning fluid, and floor mats. The monthly sales of these items at the two stores for two months are given in the following tables:

January Sales

	Vancouver	Quebec
Wiper Blades	20	15
Cleaning Fluid (bottles)	10	12
Floor Mats	8	4

February Sales

	Vancouver	Quebec
Wiper Blades	23	12
Cleaning Fluid (bottles)	8	12
Floor Mats	4	5

Use matrix arithmetic to calculate the change in sales of each product in each store from January to February.

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Solution with Technology

- To add or subtract two matrices in Excel, first input their entries in two separate arrays in the spreadsheet (we've also added labels as in the tables above, which you might do if you wanted to save the spreadsheet for later use):

	A	B	C	D
1	January Sales			
2		Vancouver	Quebec	
3	wiper blades	20	15	
4	cleaning fluid (bottles)	10	12	
5	floor mats	8	4	
6				
7	February Sales			
8		Vancouver	Quebec	
9	wiper blades	23	12	
10	cleaning fluid (bottles)	8	12	
11	floor mats	4	5	
12				
13				

- Select (highlight) a block of the same size (3×2 in this case) where you would like the answer, $F - J$, to appear, enter the formula `=B9:C11-B3:C5`, and then type Control+Shift+Enter. The easiest way to do this is as follows:

- Highlight cells B15:C17. Where you want the answer to appear
- Type `"="`.
- Highlight the matrix F . Cells B9 through C11
- Type `"-"`.
- Highlight the matrix J . Cells B3 through C5
- Type Control+Shift+Enter. Not just Enter

	A	B	C	D
1	January Sales			
2		Vancouver	Quebec	
3	wiper blades	20	15	
4	cleaning fluid (bottles)	10	12	
5	floor mats	8	4	
6				
7	February Sales			
8		Vancouver	Quebec	
9	wiper blades	23	12	
10	cleaning fluid (bottles)	8	12	
11	floor mats	4	5	
12				
13	Change in Sales			
14		Vancouver	Quebec	
15	wiper blades	=B9:C11-B3:C5		
16	cleaning fluid (bottles)			
17	floor mats			
18				
19				

Typing Control+Shift+Enter (instead of Enter) tells Excel that your formula is an *array formula*, one that returns a matrix rather than a single number.⁴⁴ Once entered, the formula bar will show the formula you entered enclosed in “curly braces,” indicating that it is an array formula. Note that you must use Control+Shift+Enter to delete any array you create: select the block you wish to delete and press Delete followed by Control+Shift+Enter.

Example 3 The revenue generated by sales in the Vancouver and Quebec branches of the A-Plus auto parts store was as follows:

January Sales in Canadian Dollars		
	Vancouver	Quebec
Wiper Blades	140.00	105.00
Cleaning Fluid (bottles)	30.00	36.00
Floor Mats	96.00	48.00

If the Canadian dollar was worth \$0.65U.S. at the time, compute the revenue in U.S. dollars.

Solution with Technology Scalar multiplication is done in Excel in much the same way we did matrix addition above:

January Sales (Can \$)		
	Vancouver	Quebec
wiper blades	140	105
cleaning fluid (bottles)	30	36
floor mats	96	48

January Sales (US \$)		
	Vancouver	Quebec
wiper blades		
cleaning fluid (bottles)		
floor mats		

As with matrix addition, we use Control+Shift+Enter to evaluate scalar multiplication (or to evaluate any formula that uses an array).

Transposition To transpose a matrix in Excel, first highlight the block where you would like the transpose to appear (shaded in the spreadsheet below). (Note that it should have the correct dimensions for the transpose: 3×2 in the case

⁴⁴ Note that on a Macintosh, Command+Enter has the same effect as Control+Shift+Enter.

shown below.) Then type the formula =TRANSPOSE (A1:C2) (use the array range appropriate for the matrix you want to transpose) in the formula bar and press Control+Shift+Enter. The easiest way to do this is as follows:

1. Highlight Cells A4:B6. The 3×2 block for the answer
2. Type “=TRANSPOSE (”.
3. Highlight the matrix you want to transpose. Cells A1:C2
4. Type “) ”.
5. Type Control+Shift+Enter. Not just Enter

The transpose will appear in the region you highlighted.

Section 3.2

Example 1 The A-Plus auto parts store had the following sales in its Vancouver store.

	Vancouver
Wiper Blades	20
Cleaning Fluid (bottles)	10
Floor Mats	8

The store sells wiper blades for \$7.00 each, cleaning fluid for \$3.00 per bottle, and floor mats for \$12.00 each. Use matrix multiplication to find the total revenue generated by sales of these items.

Solution with Technology In Excel, the formula for matrix multiplication is MMULT. (Ordinary multiplication, *, will not work.)

1. To find the product PQ of a row and a column in Excel, first enter the matrices P and Q anywhere in the spreadsheet.
2. Select the cell where you would like the result to appear (G1 in the example shown below), and use MMULT as shown. (Because the product occupies only a single cell, it is not necessary to press Control+Shift+Enter, but it won't hurt.)

- Enter the formula `=MMULT (A1 : D2 , F1 : H4)` (using the mouse to avoid typing the array ranges if you like) and press Control+Shift+Enter. The product will appear in the region you highlighted. If you try to multiply two matrices whose product is not defined, you will get the error “!VALUE#”.

	A	B	C	D	E	F	G	H	I
1	2	0	-1	3		1	1	-8	
2	1	-1	2	-2		1	0	0	
3						0	5	2	
4	-4	21	-21			-2	8	-1	
5	4	-5	-2						
6									
7									

Example 6—Identity Matrix There is no function in Excel that returns an identity matrix. If you need a small identity matrix, it's simplest to just enter the 1s and 0s by hand. If you need a large identity matrix, here is one way to get it quickly.

- Say we want a 4×4 identity matrix in the cells B1:E4. Enter the following formula in cell B1:

`=IF (ROW (B1) -ROW (B1) =COLUMN (B1) -COLUMN (B1) , 1 , 0)`

	A	B	C	D	E	F
1		=IF(ROW(B1)-ROW(\$B\$1)=COLUMN(B1)-COLUMN(\$B\$1),1,0)				
2						
3						

- Press Enter, then copy cell B1 to cells B1:E4. The formula will return 1 along the diagonal of the matrix and 0s elsewhere, giving you the identity matrix. Why does this formula work?

	A	B	C	D	E	F
1		1	0	0	0	
2		0	1	0	0	
3		0	0	1	0	
4		0	0	0	1	
5						
6						

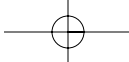
Section 3.3

Example 2 Find the inverse of

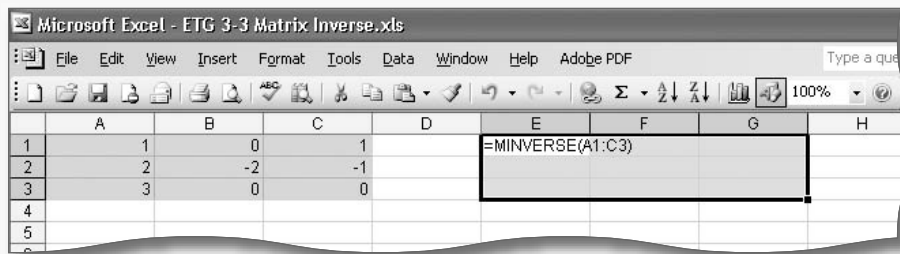
$$Q = \begin{bmatrix} 1 & 0 & 1 \\ 2 & -2 & -1 \\ 3 & 0 & 0 \end{bmatrix}$$

Solution with Technology In Excel, the function `MINVERSE` computes the inverse of a matrix.

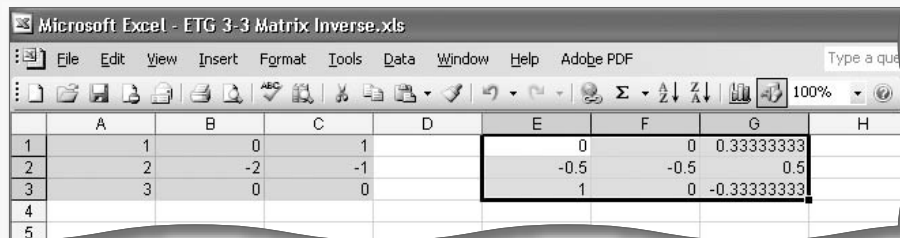
- Enter Q somewhere convenient, for example, in cells A1:C3.



2. Choose the block where you would like the inverse to appear, highlight the whole block.
3. Enter the formula $=\text{MINVERSE}(A1:C3)$ and press Control+Shift+Enter.



The inverse will appear in the region you highlighted. (To convert the answer to fractions, format the cells as fractions.)



If a matrix is singular, Excel will register an error by showing #NUM! in each cell.

Although Excel appears to invert the matrix in one step, it is going through the procedure in the text or some variation of it to find the inverse. Of course, you could also use Excel to help you go through the row reduction, just as in Chapter 2. Online, follow

Chapter 3 → Excel Tutorials → 3.3 Matrix Inversion

for a downloadable Excel tutorial that covers this and other examples in this section.

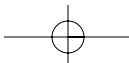
Example 4 Solve the following three systems of equations.

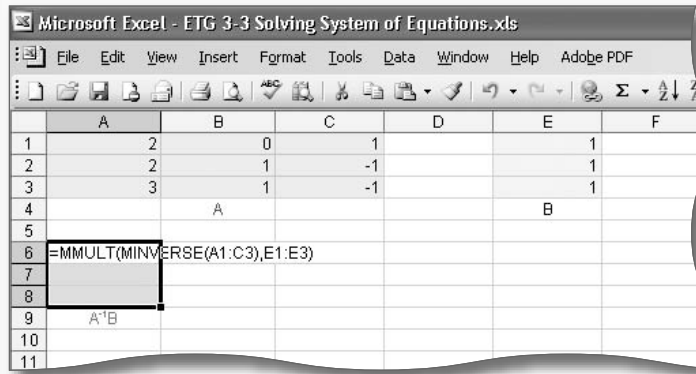
$$\begin{array}{lll}
 \text{a. } 2x & + z = 1 & \text{b. } 2x & + z = 0 & \text{c. } 2x & + z = 0 \\
 2x + y - z = 1 & & 2x + y - z = 1 & & 2x + y - z = 0 \\
 3x + y - z = 1 & & 3x + y - z = 2 & & 3x + y - z = 0
 \end{array}$$

Solution with Technology Spreadsheets like Excel instantly update calculated results every time the contents of a cell are changed. We can take advantage of this to solve the three systems of equations given above using the same worksheet as follows.

1. Enter the matrices A and B from the matrix equation $AX = B$.
2. Select a 3×1 block of cells for the matrix X .
3. The Excel formula we can use to calculate X is

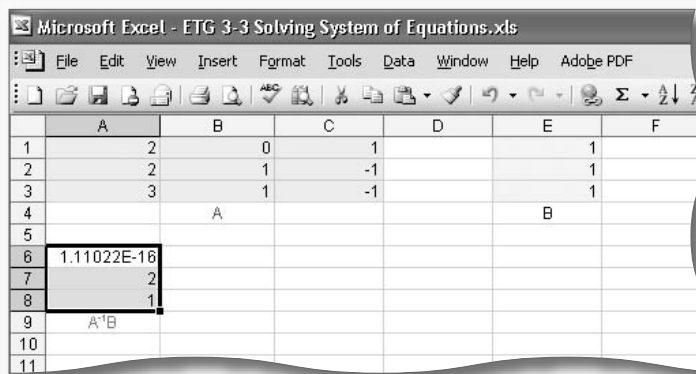
$$=\text{MMULT}(\text{MINVERSE}(A1:C3), E1:E3) \quad A^{-1}B$$





(As usual, use the mouse to select the ranges for A and B while typing the formula, and don't forget to press Control+Shift+Enter.) Having obtained the solution to part (a), you can now simply modify the entries in Column E to see the solutions for parts (b) and (c).

Note Your spreadsheet for part (a) may look like this:



What is that strange number doing in cell A6? “E-16” represents “ $\times 10^{-16}$ ”, so the entry is really

$$1.11022 \times 10^{-16} = 0.000\,000\,000\,000\,000\,111022 \approx 0$$

Mathematically, it is supposed to be *exactly* zero (see the solution to part (a) in the text) but Excel made a small error in computing the inverse of A , resulting in this spurious value. Note, however, that it is accurate (agrees with zero) to 15 decimal places! In practice, when we see numbers arise in matrix calculations that are far smaller than all the other entries, we can usually assume they are supposed to be zero. ■

Section 3.5

Example 1 Recall that the input-output table in Example 1 looks like this:

		To	Crude	Refining
From	Crude		27,000	59,000
	Refining		750	15,000
Total Output			87,000	140,000

What was the technology matrix for these two sectors? What was left over from each of these sectors for use by other parts of the economy or for export?

Solution with Technology

1. Enter the input-output table in a spreadsheet:

	A	B	C	D	E
1			To		
2			Crude	Refining	
3	From	Crude	27000	59000	
4		Refining	750	15000	
5	Total Output		87000	140000	
6					
7					

2. To obtain the technology matrix, we divide each column by the total output of its sector:

	A	B	C	D	E
1			To		
2			Crude	Refining	
3	From	Crude	27000	59000	
4		Refining	750	15000	
5	Total Output		87000	140000	
6					
7			=C3/C\$5		
8					
9					
10					

	A	B	C	D	E
1			To		
2			Crude	Refining	
3	From	Crude	27000	59000	
4		Refining	750	15000	
5	Total Output		87000	140000	
6					
7			0.31034	0.42143	
8			0.00862	0.10714	
9					
10					

The formula $=C3/C\$5$ is copied into the shaded 2×2 block shown on the left. (The $\$$ sign in front of the 5 forces the program to always divide by the total in Row 5 even when the formula is copied from Row 7 to Row 8. The result is the technology matrix shown on the right.

3. Using the techniques discussed in the second section, we can now compute $D = X - AX$ to find the demand vector.

Example 3 Consider four sectors of the economy of Kenya:⁴⁵ (1) the traditional economy, (2) agriculture, (3) manufacture of metal products and machinery, and (4) wholesale and retail trade. The input-output table for these four sectors for 1976 looks like this (all numbers are 1000s of K£):

	To	1	2	3	4
From	1	8600	0	0	0
	2	0	20,000	24	0
	3	1500	530	15,000	660
	4	810	8500	5800	2900
Total Output		87,000	530,000	110,000	180,000

⁴⁵ Figures are rounded. SOURCE: *Input-Output Tables for Kenya 1976*, Central Bureau of Statistics of the Ministry of Economic Planning and Community Affairs, Kenya.

Suppose that external demand for agriculture increased by K£50,000,000 and that external demand for metal products and machinery increased by K£10,000,000. How would production in these four sectors have to change to meet this rising demand?

Solution with Technology

1. Enter the input-output table in the spreadsheet.
2. Compute the technology matrix by dividing each column by the column total.
3. Insert the identity matrix I in preparation for the next step.

Microsoft Excel - ETG 3-5 Input-Output Models.xls

	A	B	C	D	E	F	G	H	I	J	
1	8600	0	0	0							
2	0	20000	24	0							
3	1500	530	15000	660							
4	810	8500	5800	2900							
5	87000	530000	110000	180000	Totals						
6											
7	Technology Matrix					Identity Matrix					
8	=A1/A\$5					1	0	0	0		
9						0	1	0	0		
10						0	0	1	0		
11						0	0	0	1		
12											
13											

4. To see how each sector reacts to rising external demand, you must calculate the inverse matrix $(I - A)^{-1}$, as shown below (remember to use Control+Shift+Enter each time):

Microsoft Excel - ETG 3-5 Input-Output Models.xls

	A	B	C	D	E	F	G	H	I	J	
1	8600	0	0	0							
2	0	20000	24	0							
3	1500	530	15000	660							
4	810	8500	5800	2900							
5	87000	530000	110000	180000	Totals						
6											
7	Technology Matrix					Identity Matrix					
8	0.098851	0	0	0		1	0	0	0		
9	0	0.037736	0.000218	0		0	1	0	0		
10	0.017241	0.001	0.136364	0.003667		0	0	1	0		
11	0.00931	0.016038	0.052727	0.016111		0	0	0	1		
12											
13		(I - A)				(I - A) ⁻¹					
14	=F8:I11-A8:D11					=MINVERSE(A14:D17)					
15											
16											
17											
18											
19											

5. To compute $(I - A)^{-1}D^+$, enter D^+ as a column and use the MMULT operation (see Example 3 in Section 3.2).

12									
13		(I - A)				(I - A) ⁻¹			
14	0.901149	0	0	0	1.109694	0	0	0	
15	0	0.962264	-0.000218	0	5.03E-06	1.039216	0.000263	9.79E-07	
16	-0.017241	-0.001	0.863636	-0.003667	0.022203	0.001276	1.158159	0.004316	
17	-0.00931	-0.016038	-0.052727	0.983889	0.011691	0.017008	0.062071	1.016606	
18									
19		D		(I - A) ⁻¹ D					
20		0		0					
21		50000		51963.42					
22		10000		11645.36					
23		0		1471.105					
24									
25									

Note Here is one of the beauties of spreadsheet programs: Once you are done with the calculation, you can use the spreadsheet as a template for any 4×4 input-output table by just changing the entries of the input-output matrix and/or D^+ . The rest of the computation will then be done automatically as the spreadsheet is updated. In other words, you can use it to do your homework! You can download a version of this spreadsheet by following

Chapter 3 → Excel Tutorials → 3.5 Input-Output Models ■