# Product and Quotient Rules and Higher-Order Derivatives

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# Suggested Review Topics

- Algebra skills reviews suggested:
  - Multiplying polynomials
  - Radicals as rational exponents
  - Simplifying rational expressions
  - Exponential rules
- Trigonometric skills reviews suggested:
  - Derivatives of sine and cosine

# Calculus
# Differentiation

Product and Quotient Rules and Higher-Order Derivatives

# The Product Rule

- The product of two differentiable functions $u$ and $v$ is itself differentiable.
- Moreover, the derivative of $uv$ is the first function multiplied by the derivative of the second, plus the second function multiplied by the derivative of the first.
- The formula:

$$\frac{d}{dx}[u(x)v(x)] = u(x)v'(x) + v(x)u'(x)$$

Examples: Use the product rule to find the derivative.

*1.* $f(x) = (6x + 5)(x^3 - 2)$

- Let $u(x) = 6x + 5$ and $v(x) = x^3 - 2$. Then we can find the derivatives to be $u'(x) = 6$ and $v'(x) = 3x^2$.
- Using the product rule we have

$$f'(x) = (6x + 5)(3x^2) + (x^3 - 2)(6)$$
$$= 18x^3 + 15x^2 + 6x^3 - 12$$
$$= 24x^3 + 15x^2 - 12$$

Examples: ~~Use the product rule to~~ find the derivative.

1. $f(x) = (6x+5)(x^3-2)$

Suppose we didn't use the product rule and we first multiplied the function and simplified to get

$$f(x) = 6x^4 - 12x + 5x^3 - 10$$

Then we can take the derivative and still get

$$f'(x) = 24x^3 + 15x^2 - 12$$

Either we multiply to start and take the derivative or take little derivatives and multiply at the end.

Examples: Use the product rule to find the derivative.

2. $g(x) = \sqrt{x} \sin x$

Rewrite: $g(x) = x^{1/2} \sin x$

Identify: $u(x) = x^{1/2}$ with $u'(x) = \frac{1}{2} x^{-1/2}$ and $v(x) = \sin x$ with $v'(x) = \cos x$.

Product Rule:

$$g'(x) = (x^{1/2})(\cos x) + (\sin x)(\frac{1}{2} x^{-\frac{1}{2}})$$

Rewrite: $g'(x) = \sqrt{x} \cos x + \frac{\sin x}{2\sqrt{x}}$

Examples: Use the product rule to find the derivative.

3. $h(x) = \left(x^{-2} + \frac{1}{x}\right)(\sqrt[3]{x} - \cos x)$

Rewrite: $h(x) = (x^{-2} + x^{-1})\left(x^{1/3} - \cos x\right)$

Identify:

$u(x) = (x^{-2} + x^{-1})$ $\quad u'(x) = (-2x^{-3} - x^{-2})$

$v(x) = \left(x^{1/3} - \cos x\right)$ $\quad v'(x) = (\frac{1}{3}x^{-\frac{2}{3}} + \sin x)$

Product Rule: $h'(x) = (x^{-2} + x^{-1})\left(\frac{1}{3}x^{-\frac{2}{3}} + \sin x\right) + \left(x^{1/3} - \cos x\right)(-2x^{-3} - x^{-2})$.

Examples: Use the product rule to find the derivative.

4. $y = (x^2 + 3x)(2x - 1)(x^5 - \sin x)$

The product rule can be generalized so that you take all the originals and multiply by only one derivative each time. That is, leave the first two and multiply by the derivative of the third plus leave the outside two and multiply by the derivative of the second and finally leave the last two and multiply by the derivative of the first.

Examples: Use the product rule to find the derivative.

4. $y = (x^2 + 3x)(2x - 1)(x^5 - \sin x)$

$$\begin{aligned} y' = {} & (x^2 + 3x)(2x - 1)(5x^4 - \cos x) \\ & + (x^2 + 3x)(2)(x^5 - \sin x) \\ & + (2x + 3)(2x - 1)(x^5 - \sin x) \end{aligned}$$

The same result could be found by using a nested product rule with the last two factors as your $v(x)$.

# The Quotient Rule

- The quotient $u/v$ of two differentiable functions $u$ and $v$ is itself differentiable at all values of *x* for which $v(x) \neq 0$.
- Moreover, the derivative of $u/v$ is given by the denominator multiplied by the derivative of the numerator minus the numerator multiplied by the derivative of the denominator, all divided by the square of the denominator.

# The Quotient Rule

- The rule:

$$\frac{d}{dx}\left[\frac{u(x)}{v(x)}\right] = \frac{v(x)u'(x) - u(x)v'(x)}{[v(x)]^2} \qquad v(x) \neq 0$$

- I frequently use the song type version of lo de hi minus hi de lo all over lo squared.

Examples: Use the quotient rule to find the derivative.

1. $g(t) = \frac{t^2+4}{5t-3}$

Identify:

$$u(t) = t^2 + 4 \qquad u'(t) = 2t$$
$$v(t) = 5t - 3 \qquad v'(t) = 5$$

Quotient rule:

$$g'(t) = \frac{(5t-3)(2t) - (t^2+4)(5)}{(5t-3)^2}$$

Examples: Use the quotient rule to find the derivative.

1. $g(t) = \frac{t^2+4}{5t-3}$

Quotient rule:

$$g'(t) = \frac{(5t-3)(2t) - (t^2+4)(5)}{(5t-3)^2}$$

$$= \frac{10t^2 - 6t - 5t^2 - 20}{(5t-3)^2}$$

$$= \frac{5t^2 - 6t - 20}{(5t-3)^2}$$

Examples: Use the quotient rule to find the derivative.

2. $h(x) = \frac{x}{\sqrt{x}-1}$

$$h'(x) = \frac{(\sqrt{x} - 1)(1) - (x)(\frac{1}{2}x^{-\frac{1}{2}})}{(\sqrt{x} - 1)^2}$$

$$= \frac{\sqrt{x} - 1 - \frac{x}{2\sqrt{x}}}{(\sqrt{x} - 1)^2} = \frac{\frac{1}{2}\sqrt{x} - 1}{(\sqrt{x} - 1)^2}$$

Examples: Use the quotient rule to find the derivative.

3. $f(x) = \frac{sinx}{x^3}$

$$f'(x) = \frac{x^3(cosx) - sinx(3x^2)}{(x^3)^2}$$

$$f'(x) = \frac{x^3 \cos x - 3x^2 \sin x}{x^6}$$

Examples: Use the quotient rule to find the derivative.

4. $y = \frac{x^3 - 2x^2 + 6x^{-4}}{5x^8 + \sin x}$

$$u(x) = x^3 - 2x^2 + 6x^{-4}$$
$$u'(x) = 3x^2 - 4x - 24x^{-5}$$

$v(x) = 5x^8 + \sin x \qquad v'(x) = 40x^7 + \cos x$

$$y' = \frac{(5x^8 + \sin x)(3x^2 - 4x - 24x^{-5}) - (x^3 - 2x^2 + 6x^{-4})(40x^7 + \cos x)}{(5x^8 + \sin x)^2}$$

Examples: Find the derivatives using the product and quotient rules.

1. $f(x) = (x^7 + 3x^4)(\frac{\sin x}{x})$

$$f'(x) = (x^7 + 3x^4)\frac{d}{dx}\left(\frac{\sin x}{x}\right) + (\frac{\sin x}{x})\frac{d}{dx}(x^7 + 3x^4)$$

$$f'(x) = (x^7 + 3x^4)\left[\frac{x\cos x - \sin x\,(1)}{x^2}\right] + (\frac{\sin x}{x})(7x^6 + 12x^3)$$

$$f'(x) = (x^6 + 3x^3)\cos x + (\sin x)(6x^5 + 9x^2)$$

Examples: Find the derivatives using the product and quotient rules.

2. $y = \frac{(\sin x)(3x^{-2}+5x)}{\cos x - 7}$

$$y' = \frac{(\cos x - 7)\frac{d}{dx}[(\sin x)(3x^{-2} + 5x)] - (\sin x)(3x^{-2} + 5x)\frac{d}{dx}(\cos x - 7)}{(\cos x - 7)^2}$$

$$y' = \frac{(\cos x - 7)[(\sin x)(-6x^{-3} + 5) + \cos x\,(3x^{-2} + 5x)] - (\sin x)(3x^{-2} + 5x)(-\sin x)}{(\cos x - 7)^2}$$

# Derivatives of Trigonometric Functions

- Using the quotient rule we can now find the derivatives of the remaining trigonometric functions:

$$\frac{d}{dx}[tanx] = sec^2x \qquad \frac{d}{dx}[cotx] = -csc^2x$$

$$\frac{d}{dx}[secx] = secxtanx \qquad \frac{d}{dx}[cscx] = -cscxcotx$$

- Let’s take a look at one and I’ll leave it to you to verify the others.

$$\frac{d}{dx}[tanx] = sec^2x$$

Let $y = \tan x = \frac{\sin x}{\cos x}$. Using the quotient rule we have

$$y' = \frac{\cos x\,(\cos x) - \sin x\,(-\sin x)}{(\cos x)^2}$$

Now simplify to get

$$y' = \frac{cos^2x + sin^2x}{cos^2x} = \frac{1}{cos^2x} = sec^2x$$

Examples: Find the derivatives of the trig functions.

1. $f(\theta) = (\theta + 1)\cos\theta$

$$f'(\theta) = (\theta + 1)(-\sin\theta) + \cos\theta\,(1)$$
$$f'(\theta) = \cos\theta - \sin\theta - \theta\sin\theta$$

2. $y = x + \cot x$

$$y' = 1 - csc^2x$$

Examples: Find the derivatives of the trig functions.

3. $y = \frac{\sec x}{x}$

$$y' = \frac{x(\sec x \tan x) - \sec x\,(1)}{x^2}$$

Or

$$y' = \frac{\sec x(x \tan x - 1)}{x^2}$$

Examples: Find the derivatives of the trig functions.

4. $y = x\sin x + \csc x$

$$y = x(\cos x) + \sin x\,(1) - \csc x\cot x$$

5. $h(\theta) = 5\theta\sec\theta + \theta\tan\theta$

$$h'(\theta) = 5\theta(\sec\theta\tan\theta) + \sec\theta\,(5) + \theta sec^2\theta + \tan\theta(1)$$
$$h'(\theta) = 5\theta\sec\theta\tan\theta + 5\sec\theta + \theta sec^2\theta + \tan\theta$$

Find equation(s) of the tangent line(s) to the graph of $f(x) = \frac{x+1}{x-1}$ that are parallel to the line $2y + x = 6$.

Step 1: Find slope of the given line:

$$2y = -x + 6$$

$$y = -\frac{1}{2}x + 3$$

$$m = -\frac{1}{2}$$

Step 2: Find where $f'(x) = -\frac{1}{2}$

Find equation(s) of the tangent line(s) to the graph of $f(x) = \frac{x+1}{x-1}$ that are parallel to the line $2y + x = 6$.

Step 2: Find where $f'(x) = -\frac{1}{2}$

$$f'(x) = \frac{(x-1)(1) - (x+1)(1)}{(x-1)^2}$$

$$= \frac{x - 1 - x - 1}{(x-1)^2} = \frac{-2}{(x-1)^2}$$

Solve

$$-\frac{1}{2} = \frac{-2}{(x-1)^2}$$

Find equation(s) of the tangent line(s) to the graph of $f(x) = \frac{x+1}{x-1}$ that are parallel to the line $2y + x = 6$.

Solve

$$-\frac{1}{2} = \frac{-2}{(x-1)^2}$$

$$(x-1)^2 = 4$$

$$x - 1 = \pm 2$$

$$x = -1, x = 3$$

Step 3: Find corresponding $y$ values and find equations of tangent lines.

Find equation(s) of the tangent line(s) to the graph of $f(x) = \frac{x+1}{x-1}$ that are parallel to the line $2y + x = 6$.

$$x = -1, x = 3$$

Step 3: Find corresponding $y$ values and find equations of tangent lines.

For $x = -1, y = 0$, and $y - 0 = -\frac{1}{2}(x+1)$ becomes $y = -\frac{1}{2}x - \frac{1}{2}$

For $x = 3, y = 2$, and $y - 2 = -\frac{1}{2}(x-3)$ becomes $y = -\frac{1}{2}x + \frac{7}{2}$.

# Higher Order Derivatives

- Just as we can find the derivative of a position function to find a velocity function, we can find the derivative of the velocity function (as it is just a function) to find the acceleration function.
- Though direct applications may run out after the third derivative (the jerk function), we can take derivatives as long as we want to.
- We continue to use tick marks up to the third derivative and then switch the notation to a superscript number in parentheses: $\frac{d^4y}{dx^4} = f^{(4)}(x)$.

An automobile's velocity starting from rest is $v(t) = \frac{100t}{2t+15}$ where $v$ is measured in feet per second. Find the acceleration at the following times.

a) 5 seconds

b) 10 seconds

c) 20 seconds

To find acceleration, we must find the derivative of velocity.

An automobile's velocity starting from rest is $v(t) = \frac{100t}{2t+15}$ where $v$ is measured in feet per second. Find the acceleration at the following times.

To find acceleration, we must find the derivative of velocity.

$$a(t) = v'(t) = \frac{(2t+15)(100) - 100t(2)}{(2t+15)^2}$$

$$= \frac{1500}{(2t+15)^2}$$

An automobile's velocity starting from rest is $v(t) = \frac{100t}{2t+15}$ where $v$ is measured in feet per second. Find the acceleration at the following times.

a) 5 seconds

$$a(t) = \frac{1500}{(2t + 15)^2}$$

$$a(5) = \frac{1500}{(2(5) + 15)^2} = \frac{1500}{25^2} \approx 2.4\ ft/sec^2$$

An automobile's velocity starting from rest is $v(t) = \frac{100t}{2t+15}$ where $v$ is measured in feet per second. Find the acceleration at the following times.

b) 10 seconds

$$a(t) = \frac{1500}{(2t + 15)^2}$$

$$a(10) = \frac{1500}{(2(10) + 15)^2} = \frac{1500}{35^2} \approx 1.22\ ft/sec^2$$

An automobile's velocity starting from rest is $v(t) = \frac{100t}{2t+15}$ where $v$ is measured in feet per second. Find the acceleration at the following times.

c) 20 seconds

$$a(t) = \frac{1500}{(2t + 15)^2}$$

$$a(20) = \frac{1500}{(2(20) + 15)^2} = \frac{1500}{55^2} \approx 0.50\ ft/sec^2$$

# The End