Basic Differentiation Rules and Rates of Change

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Suggested Review Topics

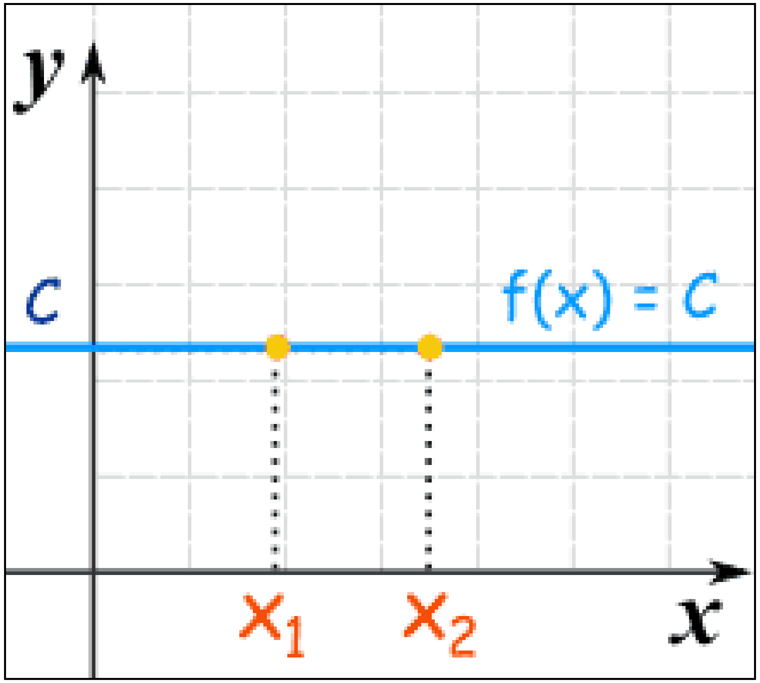
- Algebra skills reviews suggested:
  - Evaluating functions
  - Radicals as rational exponents
  - Simplifying rational expressions
  - Exponential rules
- Trigonometric skills reviews suggested:
  - Graphs of sine and cosine
  - Evaluating trigonometric functions
  - Solving trigonometric equations

Calculus
Differentiation

Basic Differentiation Rules and Rates of Change

Derivative Rules: Part 1

- **The Constant Rule** – The derivative of a constant function is 0. That is, if *c* is a real number, then $\frac{d}{dx}[c] = 0$.



Derivative Rules: Part 1

- **The Power Rule** – If *n* is a rational number, then the function $f(x) = x^n$ is differentiable and $\frac{d}{dx}[x^n] = nx^{n-1}$.

- For *f* to be differentiable at *x = 0*, *n* must be a number such that x^{n-1} is defined on an interval containing 0.

Examples: Find the slope of the tangent line at the given value.

1. $f(x) = x^5,\ x = 2$

The slope of the tangent line at the given value is the value of the derivative. Using the rules we have

$$f'(x) = 5x^{5-1} = 5x^4$$

Thus, $f'(2) = 5(2)^4 = 5(16) = 80.$

Examples: Find the slope of the tangent line at the given value.

2. $f(x) = x^{2/3}, \quad x = 1$

Using the power rule we have

$$f'(x) = \frac{2}{3}x^{\frac{2}{3}-1} = \frac{2}{3}x^{-1/3} = \frac{2}{3x^{1/3}}$$

The slope is $m = f'(1) = \frac{2}{3(1)^{1/3}} = \frac{2}{3}$.

Examples: Find the slope of the tangent line at the given value.

3. $f(x) = \frac{1}{x^3}, \; x = -2$

Rewrite: $f(x) = \frac{1}{x^3} = x^{-3}$.

Find the derivative: $f'(x) = -3x^{-3-1} = -3x^{-4}$

Rewrite: $f'(x) = \frac{-3}{x^4}$.

Evaluate the derivative: $f'(-2) = \frac{-3}{(-2)^4} = \frac{-3}{16}$

Rules Part 2

- **The Constant Multiple Rule** – If f is a differentiable function and c is a real number, then cf is also differentiable and $\frac{d}{dx}[cf(x)] = cf'(x)$.
- **The Sum and Difference Rules** – The sum (or difference) of two differentiable functions f and g is itself differentiable. Moreover, the derivative of $f + g$ (or $f - g$) is the sum (or difference) of the derivatives of f and g.
  - $\frac{d}{dx}[f(x) + g(x)] = f'(x) + g'(x)$
  - $\frac{d}{dx}[f(x) - g(x)] = f'(x) - g'(x)$

Examples: Find the derivative.

1. $f(x) = 3x^5$

$$f'(x) = 3(5x^{5-1}) = 3(5x^4) = 15x^4$$

2. $f(x) = 17x^2$

$$f'(x) = 17(2x^{2-1}) = 17(2x^1) = 34x$$

3. $f(x) = -5x^{-7}$

$$f'(x) = -5(-7x^{-7-1}) = -5(-7x^{-8}) = 35x^{-8}$$

Fact

- With constants, constant multiples and sums or differences, we can now find the derivative of any polynomial or root function using the rules rather than limits.

Examples: Use the rules of differentiation to find the derivatives of the function.

1. $f(x) = 9$

$$f'(x) = 0$$

2. $f(x) = \sqrt[4]{x}$

Rewrite: $f(x) = x^{1/4}$

Derivative: $f'(x) = \frac{1}{4}x^{\frac{1}{4}-1} = \frac{1}{4}x^{-3/4}$

Rewrite: $f'(x) = \dfrac{1}{4\sqrt[4]{x^3}}$

Examples: Use the rules of differentiation to find the derivatives of the function.

3. $f(x) = x^2 - 2x + 3$

$$f'(x) = 2x^{2-1} - 2(1x^{1-1}) + 0 = 2x - 2$$

❖Note: $x^0 = 1$ for $x \neq 0$

4. $y = 8 - x^3$

$$y' = 0 - 3x^{3-1} = -3x^2$$

Examples: Use the rules of differentiation to find the derivatives of the function.

5. $f(x) = 2x^3 - x^2 + 3x$

$$f'(x) = 2(3x^{3-1}) - 2x^{2-1} + 3(1x^{1-1})$$
$$= 6x^2 - 2x + 3$$

❖ Notice that the derivative of $y = mx$ is always m. That is, the slope of a linear equation (or linear term) is the coefficient of x.

Rules Part 3

By observing the graphs of the two basic trig functions $y = sinx$ and $y = cosx$ we find that

$$\frac{d}{dx}[sinx] = cosx \qquad \frac{d}{dx}[cosx] = -sinx$$

Examples: Use the rules of differentiation to find the derivative of the function.

1. $g(t) = \pi \cos t$

$$g'(t) = \pi(-\sin t) = -\pi \sin t$$

2. $y = 7 + \sin x$

$$y' = 0 + \cos x = \cos x$$

Examples: Use the rules of differentiation to find the derivative of the function.

3. $y = \frac{5}{(2x)^3} + 2\cos x$

Rewrite: $y = \frac{5}{8x^3} + 2\cos x = \frac{5}{8}x^{-3} + 2\cos x$

Derivative: $y' = \frac{5}{8}(-3x^{-3-1}) + 2(-\sin x)$

$$= -\frac{15}{8}x^{-4} - 2\sin x$$

Rewrite: $y' = \frac{-15}{8x^4} - 2\sin x$

Examples: Find the slope of the graph of the function at the given point.

1. $f(x) = \frac{8}{x^2}, \quad (2,2)$

Rewrite: $f(x) = 8x^{-2}$

Derivative: $f'(x) = 8(-2x^{-2-1}) = -16x^{-3}$

Rewrite: $f'(x) = \frac{-16}{x^3}$

Evaluate at (2,2): $f'(2) = \frac{-16}{(2)^3} = \frac{-16}{8} = -2$

Examples: Find the slope of the graph of the function at the given point.

2. $f(t) = 3 - \frac{3}{5t}, \quad (\frac{3}{5}, 2)$

Rewrite: $f(t) = 3 - \frac{3}{5}t^{-1}$

Derivative: $f'(t) = 0 - \frac{3}{5}(-1t^{-1-1}) = \frac{3}{5}t^{-2}$

Rewrite: $f'(t) = \frac{3}{5t^2}$

Evaluate: $f'\left(\frac{3}{5}\right) = \frac{3}{5(\frac{3}{5})^2} = \frac{3}{5(\frac{9}{25})} = \frac{3}{\frac{9}{5}} = \frac{3}{1} \cdot \frac{5}{9} = \frac{5}{3}$

Examples: Find the slope of the graph of the function at the given point.

3. $f(\theta) = 4\sin\theta - \theta, \quad (0,0)$

$$f'(\theta) = 4(\cos\theta) - 1$$

$$f'(0) = 4(\cos 0) - 1 = 4(1) - 1 = 3$$

Examples: Find the derivative

1. $f(x) = \frac{x^3-6}{x^2}$

Rewrite: $f(x) = \frac{x^3}{x^2} - \frac{6}{x^2} = x - 6x^{-2}$

Derivative: $f'(x) = 1 - 6(-2x^{-2-1}) = 1 + 12x^{-3}$

Rewrite: $f'(x) = 1 + \frac{12}{x^3}$

Examples: Find the derivative

2. $y = 3x(6x - 5x^2)$

Rewrite: $y = 18x^2 - 15x^3$

Derivative:

$$y' = 18(2x^{2-1}) - 15(3x^{3-1}) = 36x - 45x^2$$

Examples: Find the derivative

3. $y = \frac{2}{\sqrt[3]{x}} + 3\cos x$

Rewrite: $y = 2x^{-1/3} + 3\cos x$

Derivative: $y' = 2\left(-\frac{1}{3}x^{-\frac{1}{3}-1}\right) + 3(-\sin x)$

$$= \frac{-2}{3}x^{-4/3} - 3\sin x$$

Rewrite: $y' = \frac{-2}{3x^{4/3}} - 3\sin x$

Example: Determine the point(s) at which the graph of $f(x) = x + \sin x$, $0 \leq x < 2\pi$ has a horizontal tangent line.

- Think about the question. Horizontal lines have zero slope. So we are looking for points on the graph of the function where $f'(x) = 0$.

Since $f'(x) = 1 + \cos x$ we need to solve

$$0 = 1 + \cos x$$
$$-1 = \cos x$$

and on the given interval this occurs at $x = \pi$.

Example: Determine the point(s) at which the graph of $f(x) = x + \sin x$, $0 \leq x < 2\pi$ has a horizontal tangent line.

$$f'(x) = 1 + \cos x \text{ at } x = \pi$$

- We want to find the point (x, y) with $x = \pi$ so $y = f(\pi) = \pi + \sin \pi = \pi + 0 = \pi$

- Our answer is that the given function has a horizontal tangent line at the point (π, π).

Example

- The number of gallons *N* of regular unleaded gasoline sold by a gasoline station at a price of *p* dollars per gallon is given by $N = f(p)$.
  1. Describe the meaning of $f'(2.979)$
     - The derivative has units of *N* divided by units of *p* so $f'(2.979)$ is the rate of change of number of gallons sold divided by price per gallon when *p=$2.979*/gallon.
  2. Is $f'(2.979)$ usually positive or negative? Explain.
     - Usually this value is negative as stations will sell less fuel at higher prices. (The rate at which they sell fuel is negative.)

The End

Thank you to mathisfun.com for the graph of the constant function.