Math 5330 Final Exam

Name `____________________________`

1. Use the simplex method to solve

   a. max $P = 3x_1 + 6x_2 + 10x_3$
   with

$$\begin{aligned} 2x_1 + 3x_2 + 4x_3 &\leq 400 \\ 2x_1 + x_2 + 2x_3 &\leq 350 \end{aligned}$$

   and $x_1, x_2, x_3 \geq 0$

   b. Find the dual of this problem and set up the *initial* simplex tableaux for this problem, complete with artificial variables. Do not solve. Which is easier to solve, the primal (a) or dual (b)? What is the minimum of the objective for the dual problem?

2. In problem 4.2 you showed that the dual of:

$$\text{maximize } c_1x_1 + \ldots + c_Nx_N$$

with constraints

$$\begin{array}{ccccccc}
a_{1,1}x_1 & + & \ldots & + & a_{1,N}x_N & \leq & b_1, \\
\vdots & & & & \vdots & & \vdots \\
a_{k,1}x_1 & + & \ldots & + & a_{k,N}x_N & \leq & b_k, \\
a_{k+1,1}x_1 & + & \ldots & + & a_{k+1,N}x_N & = & b_{k+1}, \\
\vdots & & & & \vdots & & \vdots \\
a_{M,1}x_1 & + & \ldots & + & a_{M,N}x_N & = & b_M,
\end{array}$$

and bounds

$$\begin{array}{ccc}
x_1 & \geq & 0, \\
\vdots & & \\
x_N & \geq & 0.
\end{array}$$

was

$$\text{minimize } b_1y_1 + \ldots + b_My_M$$

with $A^Ty \geq c$, and $y_1, ..., y_k \geq 0$.

Show directly (without using the fact that these problems are duals) that $b^Ty \geq c^Tx$ for any dual feasible y and any primal feasible x. If the primal problem has an unbounded maximum, what can we say about the dual problem?

3. Consider the points $(0,0), (1,2), (3,1)$.

   a. Find the L_2 straight line $y = mx + b$ for these points.

   b. Write out a linear programming problem which, if solved, would give the L_1 line for these points. It doesn't need to be in a form that could be solved by the simplex method, and you don't need to solve it. But the constraints should be linear inequalities (or equations), they should not involve absolute values.

   c. Write out a linear programming problem which, if solved, would give the L_∞ line for these points.

4. What is the order of work for each of the following? Assume all matrices are N by N and full unless otherwise stated, and assume advantage is taken of any special structure mentioned.

a. The Jacobi method to find the eigenvalues of a symmetric matrix A.

b. Solution of $Ax = b$ using Gaussian elimination, if A is tridiagonal except A_{1N} and A_{N1} are also nonzero.

c. One LR iteration, if A is upper Hessenberg (assume no pivoting).

d. One LR iteration, if A is tridiagonal (assume no pivoting).

e. One power method iteration.

f. One inverse power method iteration (not the first iteration), assuming the LU decomposition from the first iteration is saved.

g. A Fast Fourier Transform, that is, multiplication Ax, where $A_{j,k} = exp(i2\pi(j-1)(k-1)/N)$.

h. A Slow Fourier Transform, that is, multiplication Ax using the usual matrix multiplication formula.

i. One Simplex step, for solving max $c^T x$ with $Ax \leq b, x \geq 0$, where A is M by N, and $N >> M$.

j. Solution of $Ax = b$ using Gaussian elimination, if A is banded, with bandwidth $N^{\frac{2}{3}}$.

k. The orthogonal transformation of a full matrix to a similar upper Hessenberg matrix.

5. Explain how you would find the vector x which minimizes $\|Ax - b\|_2$, if you already have the QR decomposition of the M by N matrix A. The operation count would be $O(N^\alpha)$ for what α, if we assume $M \approx 2N$? What would the operation count be if you don't have a QR decomposition?