Math 5330 Final Exam

Name ____________________________

1. a. Find the straight line $y = mx + b$ which most closely fits the data points $(0, 1), (1, 5), (2, 5)$ in the L_2 norm.

   b. Write a linear programming problem which, if solved (but don't solve it), would produce the straight line $y = mx + b$ which most closely fits these same data points in the L_∞ norm. Write the constraints in the form $Ax \geq b$. (Hint: you will have 3 unknowns, m, b and ϵ, and 3 constraints involving absolute values, which translate into 6 linear constraints.)

2. Use the simplex method to solve

a. max $P = 2x_1 + 4x_2 + x_3 + x_4$
with

$$\begin{aligned} 2x_1 + x_2 + 2x_3 + 3x_4 &\leq 12 \\ 2x_2 + x_3 + 2x_4 &\leq 20 \\ 2x_1 + x_2 + 4x_3 \qquad &\leq 16 \end{aligned}$$

and $x_1, x_2, x_3, x_4 \geq 0$
(Hint: the final basis will consist of x_1, x_2, s_3, where s_3 is the third slack variable; you can use this information to save a lot of work if you want.)

b. max $P = x_1 + x_2 + 2x_3$
with

$$\begin{aligned} x_1 + 2x_2 - x_3 &\leq 6 \\ 2x_1 + x_2 - x_3 &\leq 6 \end{aligned}$$

and $x_1, x_2, x_3 \geq 0$

3. Write the dual problem for problem 2a, and solve it. (Hint: if you use the fact that the dual solution is $y = A_b^{-T} c_b$ you can save yourself a tremendous amount of work.) Use the dual solution to guess what P_{max} for problem 2a would be if the right hand side of the second constraint were increased from 20 to 20.1.

4. What is the order of work for each of the following? Assume all matrices are N by N and full unless otherwise stated, and assume advantage is taken of any special structure mentioned.

   a. One iteration of the Jacobi method to find the eigenvalues of a symmetric matrix A.

   b. Solution of $Ax = b$ using Gaussian elimination, if A is upper Hessenberg.

   c. One QR iteration, if A is symmetric and tridiagonal.

   d. A Fast Fourier Transform, that is, multiplication Ax, where $A_{j,k} = exp(i2\pi(j-1)(k-1)/N)$.

   e. A Slow Fourier Transform, that is, multiplication Ax using the usual matrix multiplication formula.

   f. Solution of min $\|Ax - b\|_2$ using the normal equations, where A is M by N, and $M >> N$.

g. Solution of min $\|Ax - b\|_2$ using orthogonal reduction, where A is M by N, and $M >> N$.

h. One Simplex step, for solving max $c^T x$ with $Ax \leq b, x \geq 0$, where A is M by N, and $N >> M$.

i. Solution of $Ax = b$ if an LU decomposition is known.

j. One iteration of the inverse power method, for finding the smallest eigenvalue of tridiagonal matrix A.

k. Solution of $Ax = b$ using Gaussian elimination, if A is banded, with bandwidth $N^{\frac{1}{3}}$.

5. a. Write a MATLAB (or Fortran) program to efficiently solve $Ax = f$, where A is tridiagonal, and the subdiagonal, diagonal and superdiagonal are stored in vectors a, b, c respectively (see figure below). You may assume pivoting is not necessary.

function x = trid(a,b,c,N,f)

$$\begin{bmatrix} b_1 & c_1 & 0 & 0 & 0 & 0 \\ a_2 & b_2 & c_2 & 0 & 0 & 0 \\ . & . & . & . & . & . \\ 0 & a_k & b_k & c_k & 0 & 0 \\ 0 & 0 & a_{k+1} & b_{k+1} & c_{k+1} & 0 \\ . & . & . & . & . & . \\ 0 & 0 & 0 & a_{N-1} & b_{N-1} & c_{N-1} \\ 0 & 0 & 0 & 0 & a_N & b_N \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ x_k \\ x_{k+1} \\ . \\ x_{N-1} \\ x_N \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \\ . \\ f_k \\ f_{k+1} \\ . \\ f_{N-1} \\ f_N \end{bmatrix}$$

b. If pivoting is done, describe how your program would change, in general terms (no need to write a new program).