

Math 5330, Test I

Name _____

1. Find the LU decomposition (no pivoting) of

$$A = \begin{bmatrix} 1 & 0 & -2 \\ -2 & -3 & 5 \\ 3 & 9 & -4 \end{bmatrix}$$

2. A MATLAB program to solve a symmetric system $Ax = b$ does most of its work in the loops:

```
for I=1:N-1
    for J=I+1:N
        for K=J:N
            A(J,K) = A(J,K) - LJI*A(I,K)
        end
    end
end
```

For large N , approximately how many multiplications are done (show work)?

3. Prove that $\frac{\|\Delta x\|}{\|x\|} \leq \text{cond}(A) \frac{\|\Delta b\|}{\|b\|}$ if $Ax = b$ and $A(x + \Delta x) = b + \Delta b$.

4. If we use the usual finite difference approximation, the DE $u''(x) = f(x), u(0) = u(\pi) = 0$ becomes:

$$\begin{aligned} U_{i+1} - 2U_i + U_{i-1} &= h^2 f(x_i), \quad i = 1, \dots, N-1 \\ U(x_0) &= U(x_N) = 0 \end{aligned}$$

where $h = \pi/N, x_i = ih, U_i \approx u(x_i)$.

a. This is a linear system of $N-1$ equations for the $N-1$ unknowns U_1, \dots, U_{N-1} . If a band solver is used to solve the system, the work is proportional to what power of N ?

b. If Jacobi's iterative method is used to solve it, the iteration will take the form $U^{k+1} = BU^k + c$; what is the matrix B ?

- c. What are the eigenvalues of the B matrix (hint: for any $m = 1, \dots, N - 1$, the vector U with components $U_i = \sin(mx_i)$ is an eigenvector. You will need the trig identity $\sin(a + b) = \sin(a)\cos(b) + \cos(a)\sin(b)$)
- d. What is the largest eigenvalue of B in absolute value? Will the Jacobi method converge?
- e. Given that the error goes down each iteration by a factor approximately equal to the largest eigenvalue, estimate how many iterations of the Jacobi method are required to decrease the error by a factor of ϵ . (Hint: $\cos(z) \approx 1 - z^2/2$ and $\ln(1 + z) \approx z$ for $z \approx 0$)
- f. The total work to solve the linear system using the Jacobi iterative method is then proportional to what power of N ? Which is faster for this tridiagonal system—a band solver or the Jacobi iterative method?

- g. If the Gauss-Seidel iterative method is used to solve the linear system, what is the matrix B (see part (b)) now? You need not write the matrix out explicitly, for example, you can write it as $E^{-1}F$, where you define E and F . Gauss-Seidel will converge if and only if what is true about B ?