Math 5330, Test I

Name _____

1. If

$$A = \begin{bmatrix} 0 & 3 & 1 \\ -4 & 2 & 1 \\ 8 & 2 & 3 \end{bmatrix}$$

find a permutation matrix P, a lower triangular matrix L, and an upper triangular matrix U such that $A = PLU$.

2. An N by N band matrix has K non-zero diagonals below the main diagonal and L above. If $1 << K, L << N$, approximately how many multiplications are done:

   a. during the forward elimination, if no pivoting is done?

   b. during the forward elimination, if partial pivoting is done?

   c. during back substitution, if no pivoting is done?

   d. during back substitution, if partial pivoting is done?

3. a. Prove that the Jacobi method:

$$x_i^{n+1} = \frac{1}{a_{ii}} \left(b_i - \sum_{j \neq i} a_{ij} x_j^n \right)$$

converges, if A is diagonal dominant.

b. Prove that the Gauss-Seidel method:

$$x_i^{n+1} = \frac{1}{a_{ii}} \left(b_i - \sum_{j<i} a_{ij} x_j^{n+1} - \sum_{j>i} a_{ij} x_j^n \right).$$

converges, if A is diagonal dominant.

4. Which of the following linear systems would you expect to produce the most relative round-off error, using Gauss elimination with partial pivoting? Justify your answer.

$$\begin{bmatrix} 10^{-9} & 10^{-8} \\ 10^{-8} & 10^{-9} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 1000 & 1001 \\ -999 & -1000 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 10^{-9} & 0 \\ 0 & 10^{9} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

5. Define:

   a. orthogonal matrix

   b. lower Hessenberg matrix

   c. positive definite matrix

   d. $\|x\|_p$, if x is a vector and $1 \leq p < \infty$

   e. $\|A\|_p$, if A is a matrix

6. The following Fortran program solves a linear system $Ax = b$ with symmetric matrix A, using Gauss-Jordan without pivoting, but taking advantage of symmetry. For large N, approximately how many multiplications are done? Show your work.

```
      SUBROUTINE DLINEQ(A,N,X,B)
      DOUBLE PRECISION A(N,N),X(N),B(N),LJI
C                     REDUCTION TO DIAGONAL
      DO 50 I=1,N
C                     ELIMINATE ELEMENTS ABOVE DIAGONAL IN COLUMN I
         DO 20 J=1,I-1
            LJI = A(J,I)/A(I,I)
            DO 10 K=I,N
               A(J,K) = A(J,K) - LJI*A(I,K)
   10       CONTINUE
            B(J) = B(J) - LJI*B(I)
   20    CONTINUE
C                     ELIMINATE ELEMENTS BELOW DIAGONAL IN COLUMN I.
C                     TAKE ADVANTAGE OF SYMMETRY HERE.
         DO 40 J=I+1,N
            LJI = A(I,J)/A(I,I)
            DO 30 K=J,N
               A(J,K) = A(J,K) - LJI*A(I,K)
   30       CONTINUE
            B(J) = B(J) - LJI*B(I)
   40    CONTINUE
   50 CONTINUE
C                     SOLVE DIAGONAL SYSTEM
      DO 55 I=1,N
         X(I) = B(I)/A(I,I)
   55 CONTINUE
      RETURN
      END
```