Num Integration

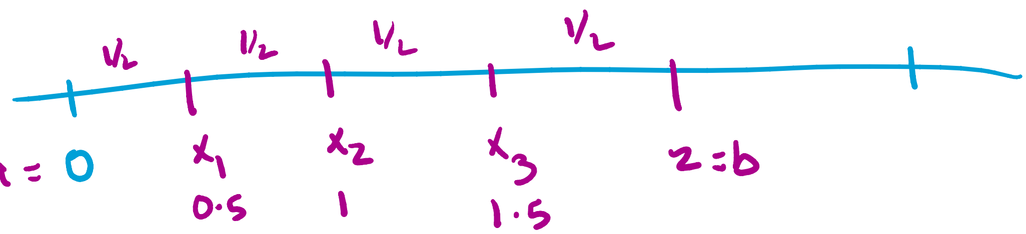
Tuesday, October 22, 2019 12:00 PM

$f(x) = 1/1+x^2 \quad 0 \le x \le 2$

Approximate $I = \int_0^2 \frac{1}{1+x^2}\,dx$

using $T_n(f)$ and $S_n(f)$ and we want to find n such that

$$\left| \int_0^2 \frac{dx}{1+x^2} - T_n(f) \right| < 5 \times 10^{-6}.$$



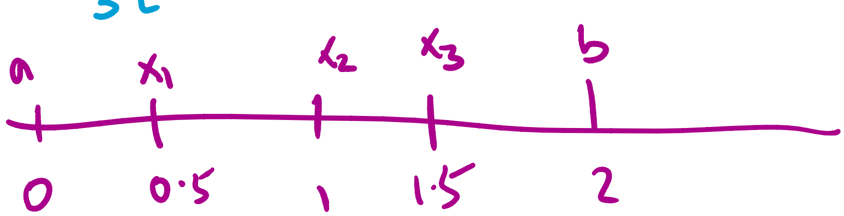
$$T_n(f) = \frac{h}{2}\left[f(a) + 2f(x_1) + \dots + 2f(x_{n-1}) + f(b) \right]$$

$$h = \frac{b-a}{n}$$

$\boxed{n=4} \rightarrow T_4(f) \qquad h = \frac{2-0}{4} = 1/2$

$$T_4(f) = \frac{0.5}{2}\left[f(0) + 2f(0.5) + 2f(1) + 2f(1.5) + f(2) \right]$$

$$S_n(f) = \frac{h}{3}\left[f(a) + 4f(x_1) + 2f(x_2) + \dots + 4f(x_{n-1}) + f(b) \right]$$



$$h = \frac{b-a}{n}$$

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$$S_4(f) = \frac{0.5}{3}\left[f(0) + 4f(0.5) + 2f(1) + 4f(1.5) + f(2)\right]$$

Error Representation:

How large should n be so that

$$\left| \int_0^2 1/1+x^2 \, dx - T_n(f) \right| < 5 \times 10^{-6} ?$$

$$f(x) = 1/1+x^2$$

Recall the error Representation formula:

$$\int_a^b f(x)\,dx - T_n(f) = -\frac{(b-a)}{12} h^2 f''(c_n)$$

where c_n is unknown number bet. a & b.

$$h = \frac{b-a}{n}.$$

Pre Requisite: $f''(c_n)$

Second Derivative Test:

$f'(x^*) = 0$ & $f''(x^*) < 0 \Rightarrow$

$f(x) \leq f(x^*)$ $a \leq x \leq b$.

$$f(x) = 1/1+x^2$$

$$f'(x) = -\frac{1}{(1+x^2)^2} * 2x \quad \text{(chain Rule)}$$

$$f''(x) = -\left(\frac{2(1+x^2)^2 - 2(1+x^2)(2x) * 2x}{(1+x^2)^4}\right) \quad \text{(Quotient Rule)}$$

$$f''(x) = -\left(\frac{2(1+x^2)^2 - 2(1+x^2)(2x)*2x}{(1+x^2)^4} \right) \quad \text{Rule)}$$

$$= -2(1+x^2)\left(\frac{(1+x^2) - 4x^2}{(1+x^2)^4} \right)$$

$$f''(x) = \frac{-2}{(1+x^2)^3}(1-3x^2) = \frac{6x^2-2}{(1+x^2)^3}$$

$$\boxed{\int_0^2 \frac{dx}{1+x^2} - T_n(f)} = -h\,\frac{(2-0)}{12}\,\frac{(6c_n^2-2)}{(1+c_n^2)^3} \qquad 0 \le c_n \le 2.$$

$$h = \frac{2-0}{n}$$

$c_n = ?$ gives max ?

$c_n = 0$ gives max value

$$E_n^T(f) = -h\left(\frac{2}{12}\right)\left(\frac{-2}{1}\right)$$

$$= \frac{h}{3}$$

find n:

$$|E_n^T(f)| < 5\times10^{-6}$$

$$\left(\frac{2}{n}\right)\frac{1}{3} < 5\times10^{-6} \quad \text{because } h = \frac{2}{n}$$

$$516.4 \approx \frac{2}{3}\times\frac{1}{5\times10^{-6}} < n \Rightarrow \boxed{n \ge 517.}$$ answer.

Numerical Integration :

$$I = \int_a^b f(x)\,dx \qquad a = x_0, \; x_1, \; x_2, \; x_3 \ldots, \; x_n = b$$

$$T_n(f) \rightarrow w_0 f(x_0) + w_1 f(x_1) + \ldots + w_n f(x_n)$$

$S_n(f)$ ↗

$I_n(f)$ ...

$S_n(f)$

Trapezoidal $\to$ $\underline{w_0} = h/2 = w_n$, $w_1 = w_2 \ldots = w_{n-1} = h$

WEIGHTS

Simpson $\to$ $\underline{w_0} = h/3 = \underline{w_n}$, $\underline{w_1} = \frac{4h}{3}$ $\underline{w_2} = \frac{2h}{3} \ldots$

$x_i \to$ nodes.

We want to DESIGN $\underbrace{\text{numerical integration}}_{\text{Quadrature}}$ formulas /rules.

Gaussian Quadrature formula:

-1 $\quad$ 1

$$\int_{-1}^{1} f(x)\,dx \simeq \tilde{I}_2(f) \to \text{2 point Gaussian Quadrature Rule.}$$

$$= w_0 f(x_0) + w_1 f(x_1)$$

$-1 = a$ $\quad$ $b = 1$

Based on the given Gaussian Quad. Table

n	x_i	w_i
2	± 0.5773	1

(Table 5.7, Textbook)

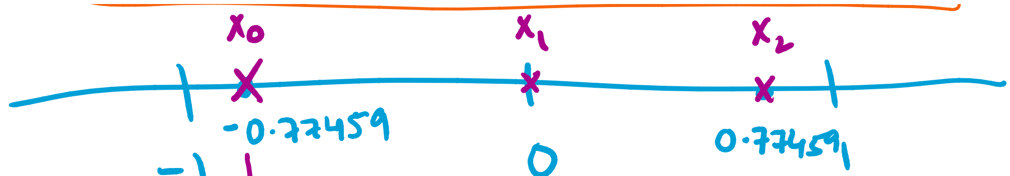
$$\tilde{I}_2(f) = 1 * f(-0.5773) + 1 * f(0.5773)$$

Example: Use the 3 point Gaussian Quadrature Rule to approximate $\int_{-1}^{1} \frac{dx}{1+x^2}$.

Table 5.7 (Textbook)

n	x_i	w_i
3	± 0.77459	0.5555
	0	0.8888

x_0 $\quad$ x_1 $\quad$ x_2



$w_0 = 0.5555$ $\quad$ $w_2 = 0.5555$

$$\boxed{\underset{w_1}{0.8888}}$$

$$f(x) = \frac{1}{1+x^2}$$

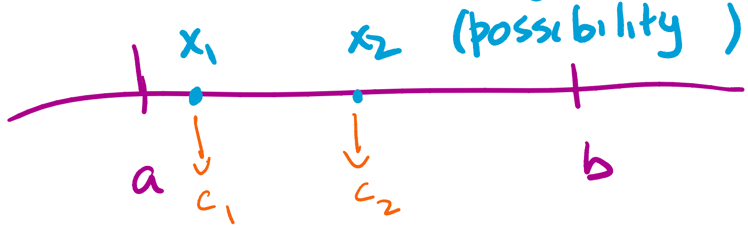
$$\hat{I}_3(f) = \frac{\overset{w_0 f(x_0)}{0.5555}}{1+(\underbrace{-0.77459}_{x_0})^2} + \frac{\overset{w_1 f(x_1)}{0.8888 \times 1}}{1+\underbrace{0^2}_{x_1}} + \frac{\overset{w_2 f(x_2)}{0.5555}}{1+(0.77459)^2}$$

Simply & obtain the approxm!

How to design my own Quadrature Rule?

$$I = \int_a^b f(x)\,dx$$

$$\approx c_1 f(x_1) + c_2 f(x_2) \quad \text{(one possibility)}$$



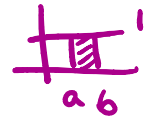
$\boxed{\text{Demand}}$ certain accuracy from this Quadrature Rule!

Determine (c_1, x_1) and (c_2, x_2) so that

$$\hat{I}(f) \rightarrow \boxed{c_1 f(x_1) + c_2 f(x_2)} = \int_a^b f(x)\,dx = I(f)$$

of $f(x)$.

for certain choices of $f(x)$.



$\boxed{f(x) = 1} \rightarrow \hat{I}(f) = I(f) = \int_a^b 1\,dx = b-a$

$\boxed{f(x) = x} \rightarrow \hat{I}(f) = I(f) = \int_a^b x\,dx = \frac{1}{2}(b^2-a^2)$

$\boxed{f(x) = x^2} \rightarrow \hat{I}(f) = I(f) = \int_a^b x^2\,dx = \frac{b^3-a^3}{3}$

$\boxed{f(x^3) = x^3} \rightarrow \hat{I}(f) = I(f) = \int_a^b x^3\,dx = \frac{1}{4}(b^4-a^4)$

$\hat{I}(f) = c_1 f(x_1) + c_2 f(x_2)$

$f(x) = 1 \rightarrow c_1 + c_2 = b-a$

$f(x) = x \rightarrow c_1 x_1 + c_2 x_2 = \frac{b^2-a^2}{2}$

$f(x) = x^2$ and $f(x) = x^3$.

Note: If $a = -1$ and $b = 1$ then

$c_1 = w_0$ & $c_2 = w_1$ from 2 pt Gaussian Quad. form

$x_1 = -0.5773$ $x_2 = +0.5773$