4329-I

Thursday, October 24, 2019 11:59 AM

Last Lecture: $I = \int_a^b f(x)\,dx$

Gaussian Quadrature formula:

$$\int_{-1}^{1} f(x)\,dx \simeq w_1 f(x_1) + w_2 f(x_2)$$

According to Table 5.7 2-point G'Quad.

$w_1 = w_2 = 1$

$x_1 = -0.5773 \quad x_2 = 0.5773$

$\hat{I}_2(f)$ = Gaussian Quad formula

$= f(-0.5773) + f(0.5773)$

3(d) (Review) $\int_{-1}^{1} x^8\,dx$ use 2-pt Gaussian Quad. Rule

Given info:

$$\boxed{\begin{array}{ll} x_1 = -0.5773 = -\frac{1}{\sqrt{3}} & w_1 = w_2 = 1 \\ x_2 = 0.5773 = \frac{1}{\sqrt{3}} & \end{array}}$$

$f(x) = x^8$

$$\hat{I}_2(f) = \left(-\frac{1}{\sqrt{3}}\right)^8 + \left(\frac{1}{\sqrt{3}}\right)^8$$

$$\hat{I}_2(f) = \frac{1}{81} + \frac{1}{81} = \frac{2}{81}$$

Compute $I = \int_{-1}^{1} x^8\,dx = \frac{2}{9}$ (check!)

error $= I - \hat{I}_2(f)$

$= 2/9 - 2/81 = 16/81$

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$$\tilde{I}(f) = \boxed{c_1 f(x_1) + c_2 f(x_2)}$$

Goal: $I(f) = \int_0^1 f(x)\,dx \approx c_1 f(0) + c_2 f(1)$

$x_1 = 0 \qquad x_2 = 1$

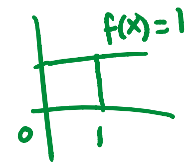
find c_1 and c_2 so that the integration formula: $\boxed{c_1 f(0) + c_2 f(1) = \tilde{I}(f)}$

is exact for $f(x) = 1$ and $f(x) = x$.

$\tilde{I}(f)$ is exact for $f(x) = 1 \Rightarrow$

$\tilde{I}(1): c_1 * 1 + c_2 * 1 = \int_0^1 1\,dx$

$$\boxed{c_1 + c_2 = \int_0^1 1\,dx = 1}$$



$\tilde{I}(f)$ is exact for $f(x) = x$

that is $\tilde{I}(x) = I(x) = \int_0^1 x\,dx$

$c_1 * 0 + c_2 * 1 = \frac{1}{2}$

$$\boxed{c_2 = 1/2}$$

$$\boxed{c_1 = 1 - 1/2 = 1/2}$$

Paraphrasing: used $\tilde{I}(f) = I(f)$ for $f(x) = 1$ and $f(x) = x$.

that gives us 2 eqns:

$c_1 + c_2 = 1$

$c_2 = 1/2$

So we get $c_1 = 1/2$ & $c_2 = 1/2$.

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$$\int_0^1 f(x)\,dx \approx \frac{1}{2} f(0) + \frac{1}{2} f(1)$$

$$\boxed{c_1 f(0) + c_2 f(1)}$$

Note: $\underbrace{\hat{I}(f) = c_1 f(0) + c_2 f(1)}_{} \approx \int_0^1 f(x)\,dx = I(f)$

Demanded: $\boxed{\rightarrow \hat{I}(f) = I(f) \quad f(x)=1 \text{ and } f(x)=x}$

$\Rightarrow$ If $f(x)$ = poly. of degree 1

then $\hat{I}(f) = I(f)$

$\boxed{\text{Int. formula is exact for poly. of degree 1.}}$

Degree of Precision of the int. formula

Our int. formula has DoP (Degree of precision) 1

Review Ques 3(a) find c_1 & c_2 in the foll. quad formula:

$$I(f) = \int_{-1}^{1} f(x)\,dx \approx \underbrace{c_1 f(-1) + c_2 f(1)}_{\hat{I}(f)}$$

So that it is exact for all polynomials of degree AT MOST 1.

(Equivalently, DoP of $\hat{I}(f)$ is 1)

$\underbrace{f(x)=1}$ $\Rightarrow$ $\hat{I}(f) = I(f) := \int_{-1}^{1} 1\,dx = 2$

$f(-1)=1$

$f(1)=1$

$$\int_{-1}^{1} 1\,dx = x\Big|_{x=-1}^{1} = 1-(-1) = 2$$

$c_1 f(-1) + c_2 f(1) = 2$

$$c_1 f(-1) + c_2 f(1) = 2$$
$$\boxed{c_1 + c_2 = 2}$$

$f(x) = x \Rightarrow c_1 f(-1) + c_2 f(1) = \int_{-1}^{1} x\,dx = \frac{x^2}{2}\Big|_{-1}^{1}$

$\left.\begin{array}{l} f(-1) = -1 \\ f(1) = 1 \end{array}\right\} \rightarrow \boxed{-c_1 + c_2 = 0}$ $\quad = \frac{1}{2} - \frac{1}{2} = 0$

$$c_1 + c_2 = 2$$
$$c_1 = c_2$$
$$c_1 = c_2 = 1$$

$$I(f) = \int_{-1}^{1} f(x)\,dx \approx \underbrace{c_1 f(-1) + c_2 f(1)}_{\tilde{I}(f)}$$

$\tilde{I}(f) = I(f)$ $\quad f(x) = 1$ & $f(x) = x$

$$\int_{-1}^{1} f(x)\,dx \approx c_1 f(\underset{x_1}{-1}) + c_2 f(\overset{x_2}{0}) = \tilde{I}(f)$$

Different node used!

Remark: $[x_1 = -0.577 \quad x_2 = 0.577]$

Find c_1 & c_2 so that $\tilde{I}(f)$ is exact for all $f(x)$ being poly. of degree at most 1.

$f(x) = 1 \Rightarrow c_1 f(-1) + c_2 f(0) = \int_{-1}^{1} 1\,dx = 2$

$f(-1) = 1$, $f(0) = 1$ $\quad \boxed{c_1 + c_2 = 2}$

$f(x) = x \Rightarrow c_1 f(-1) + c_2 f(0) = \int_{-1}^{1} x\,dx = 0$

$$-c_1 + c_2 * 0 = 0$$

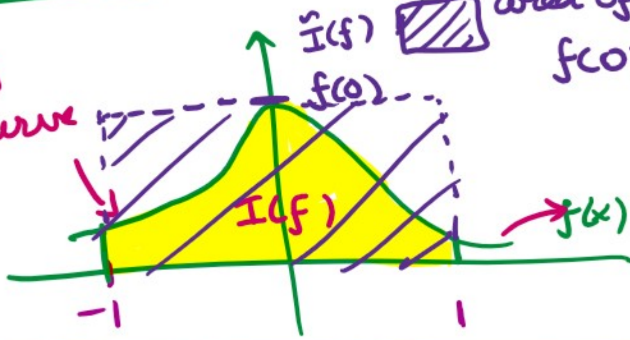
$$\boxed{c_1 = 0}$$

$$\boxed{c_1 = 0}$$

$c_1 = 0$ & $c_2 = 2$

$$\boxed{\int_{-1}^{1} f(x)\,dx \approx \underbrace{2 f(0)}_{\boxed{\tilde{I}(f)}}}$$

Suppose $f(x)$ is the foll. curve



area of rectangle width 2 height $f(0)$.

$f(0) * 2$
↑
length of interval

<u>Review</u>: Exclude #4 include polynomial interpolation

* Poly. Interpolation (Lagrange & Divided Diff formula)
* Error in poly interpolation

(Formulas for poly & error formula need to be memorized)

<u>Ques</u>; Find the poly. interpolating:

$\underset{x_0\ y_0}{(-1, 1)}$, $\underset{x_1\ y_1}{(0, 2)}$, $\underset{x_2\ y_2}{(1, 4)}$ $\quad n = 2$

$n+1$ data pts given

using <u>Lagrange formula</u> and verify that it is the same poly. as the one using Newton's D.D formula.

$$p_2^L(x) = y_0 L_0(x) + y_1 L_1(x) + y_2 L_2(x)$$

$P_2^L(x) = y_0 L_0(x) \ldots y_1 \ldots =$

$$= \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + 2\,\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + 4\,\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

$$= \frac{x(x-1)}{(-1)(-2)} + 2\,\frac{(x+1)(x-1)}{(0+1)(0-1)} + \overset{2}{\cancel{4}}\,\frac{(x+1)x}{\underset{2}{\cancel{(1+1)}}}$$

$$\boxed{\begin{aligned} P_2^L(x) &= \frac{x(x-1)}{2} - 2(x^2-1) + 2x(x+1) \\ &= (x^2-x)/2 \quad -2x^2+2 \quad + 2x^2+2x \end{aligned}}$$

$$P_2^N(x) = \underset{y_0}{f(x_0)} + \underset{1}{f[x_0,x_1]}(x-x_0) + \boxed{f[x_0,x_1,x_2]}(x-x_0)(x-x_1)$$

x_i	$y_i = f(x_i)$	$f[x_i,x_{i+1}]$
-1	1	
		$\frac{2-1}{0-(-1)} = 1$
0	2	
		$\frac{4-2}{1-0} = 2$
1	4	

$$f[x_0,x_1,x_2] = \frac{f[x_1,x_2]-f[x_0,x_1]}{x_2-x_0}$$

$$\frac{2-1}{1-(-1)} = \frac{1}{2}$$

$$P_2^N(x) = y_0 + 1\,(x-x_0) + \tfrac{1}{2}(x-x_0)(x-x_1)$$

$$= 1 + (x+1) + \underbrace{\tfrac{1}{2}(x+1)(x-0)}_{\frac{x^2+x}{2}}$$

$$= \frac{x^2}{2} + \frac{x}{2} + x + 2$$

$$\boxed{P_2^N(x) = \frac{x^2}{2} + \frac{3}{2}x + 2}$$

check: $(-1,1)$
$(0,2)$
$(1,4)$

same as $P_2^L(x) = \dfrac{x^2-x}{2} + 2 + 2x$

$$\boxed{P_2^L(x) = \frac{x^2}{2} + \frac{3x}{2} + 2}$$

#2 Without calculating p(x) estimate error in interpolating $\underbrace{\sin(\pi x)}_{f(x)}$

$\boxed{\begin{array}{c} x = 0, 0.5, 1, -1 \\ \text{4 data points} \end{array}}$ $\Rightarrow p_3(x)$

$|f(x) - p_3(x)|$ (Similar to taylor Remainder)

$n=3$

$$\frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(c_x)$$

same

$$= \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{4!} f^{(IV)}(c_x)$$

fourth order deri

c_x = unknown bet. -1 and 1.

$f(x) = \sin \pi x$

$f'(x) = \pi \cos \pi x$

$f''(x) = -\pi^2 \sin \pi x$

$f'''(x) = -\pi^3 \cos \pi x$

$f^{(IV)}(x) = \pi^4 \sin \pi x$

$$|f(x) - p_3(x)| = \left| \frac{(x-0)(x-0.5)(x-1)(x+1)}{4!} \pi^4 \sin \pi c_x \right|$$

$-1 \le x \le 1$

$c_x = 0.5$

$\sin \pi/2 = 1$

$$\le \frac{1}{24} |x(x-0.5)(x-1)(x+1)| \pi^4 * 1$$

$$= \frac{\pi^4}{24} |x(x-0.5)(x-1)(x+1)|$$

given $-1 \le x \le 1$

$|x| \le 1$

$|x - 0.5| \le 1.5$ at $x = -1$

$|x-1| \le 1$ $x = 0$

$x = 1$

$$|x-1| \leq 1 \qquad x=0$$
$$|x+1| \leq 2 \qquad x=1$$

$$\leq \frac{\pi^4}{24} \; 1 * 1.5 * 1 * 2 = \frac{\pi^4}{8}$$

$$f(x) = e^x \cos 4x \qquad -\pi \leq x \leq \pi$$

3(b) What is the error in using $T_4(f)$ & $S_4(f)$ without computing $T_4(f)$ and $S_4(f)$.

Error Formula will be provided!

$$I = \int_{-\pi}^{\pi} e^x \cos 4x \, dx \approx T_4(f)$$

$$\downarrow$$

$$S_4(f)$$

$$I - T_4(f) = \frac{-(b-a)h^2}{12} f''(c) \quad \text{(given in exam)}$$

$$h = \frac{b-a}{n}, \; n=4.$$

$$I - S_4(f) = \frac{-(b-a)h^4}{180} f''''(c)$$

$$f(x) = e^x \cos 4x$$

$f''(x)$ and $f''''(c)$