Math 4329-II

Thursday, October 24, 2019 1:26 PM

$$\hat{I}(f) = c_1 f(x_1) + c_2 f(x_2)$$

$f(x)$ integrand $\int_a^b f(x)\,dx$

example: #3(a) Review

use 2 pt Gaussian Quad formula (Given)

$$\int_{-1}^{1} f(x)\,dx \approx c_1 f(x_1) + c_2 f(x_2)$$

$c_1 = 1$ $c_2 = 1$

$x_1 = -1/\sqrt{3}$ $x_2 = 1/\sqrt{3} \approx 0.5773$

≈ -0.5773

to approximate $\int_{-1}^{1} x^8\,dx$

$f(x) = x^8$

$$\hat{I}(f) = f(-1/\sqrt{3}) + f(1/\sqrt{3})$$

$$= \left(-1/\sqrt{3}\right)^8 + \left(1/\sqrt{3}\right)^8$$

$$= 2 \; 1/(\sqrt{3})^8$$

$$\hat{I}(f) = 2/3^{8/2} = 2/3^4 = 2/81$$

Calculate $I(f) = \int_{-1}^{1} x^8 dx$ directly and compute

$I(f) - \hat{I}(f)$

$$\int_{-1}^{1} x^8\,dx = 2/9 \text{ (verify!)}$$

error $I(f) - \hat{I}(f) = \frac{2}{9} - \frac{2}{81} = \frac{16}{81}$

Goal: Construct a numerical int. formula.

$$I(f) = \int_0^1 f(x)\,dx \approx c_1 f(0) + c_2 f(1) = \hat{I}(f)$$

$0 \to x_1$, $1 \to x_2$

for any integrand $f(x)$

for any integrand $f(x)$ $\quad x_1 \quad x_2$

find c_1 and c_2 so that:
$\tilde{I}(f)$ is exact for $f(x)=1$ and $f(x)=x$.

$\tilde{I}(f) = I(f)$ when $f(x)=1 \rightarrow$ (1)
$\tilde{I}(f) = I(f)$ when $f(x)=x \rightarrow$ (2)

(1)' $\rightarrow \boxed{f(x)=1}$ $\quad c_1 f(0) + c_2 f(1) = \int_0^1 f(x)\,dx$
$c_1 + c_2 = \int_0^1 1\,dx = x\Big|_{x=0}^{1} = 1$

(2)' $\rightarrow f(x)=x$ $\quad c_1 f(0) + c_2 f(1) = \int_0^1 f(x)\,dx$
$f(0)=0 \quad f(1)=1$
$c_1 * 0 + c_2$
$\int_0^1 x\,dx = \frac{x^2}{2}\Big|_0^1 = 1/2$

(1)' and (2)' $\Rightarrow$
$$\left.\begin{aligned} c_1 + c_2 &= 1 \\ c_2 &= 1/2 \end{aligned}\right\} \Rightarrow \boxed{c_1 = 1/2}$$

$\tilde{I}(f) = c_1 f(0) + c_2 f(1)$ using $\tilde{I}(f) = I(f)$
$f(x)=1$ and
$f(x)=x$

gives me $c_1 = c_2 = 1/2$.

$$\tilde{I}(f) = \frac{f(0)}{2} + \frac{f(1)}{2}.$$

$\Rightarrow$ If $f(x) = a+bx$ for any a & b then
$$\tilde{I}(f) = \frac{f(0)}{2} + \frac{f(1)}{2} = \frac{a}{2} + \frac{a+b}{2} = a + b/2$$
$$I(f) = \int_0^1 (a+bx)\,dx = ax + \frac{bx^2}{2}\Big|_{x=0}^{1} = a + b/2$$

$\Rightarrow \tilde{I}(f) = I(f)$ $\quad$ $\tilde{I}(f)$ is exact for

$\Rightarrow \tilde{I}(f) = I(f)$ for $f(x) = 1$ and $f(x) = x$ $\Big\} \Rightarrow \tilde{I}(f)$ is exact for all linear poly.

$$\tilde{I}(f) = c_1 f(x_1) + c_2 f(x_2) + c_3 f(x_3)$$

is $\boxed{\text{exact}}$ for $f(x)$ being poly of degree 1
degree 2

exactness of integration formula $\tilde{I}(f)$
is known as the DoP (DEGREE OF PRECISION)

$$\boxed{\tilde{I}(f) = \frac{f(0)}{2} + \frac{f(1)}{2}} \approx \int_0^1 f(x)\,dx$$

Check if $\tilde{I}(f)$ is exact for $\boxed{f(x) = x^2}$?

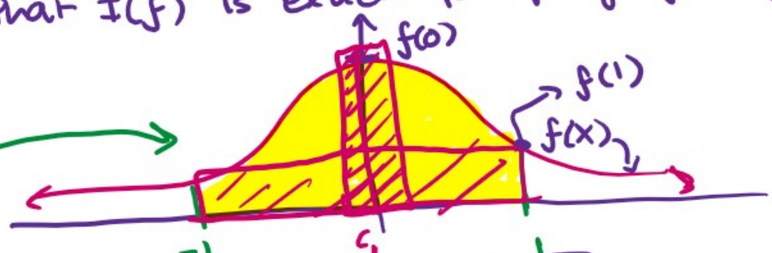
$$\left.\begin{aligned} \tilde{I}(f) &= \tfrac{1}{2}0^2 + \tfrac{1}{2}1^2 = 1/2 \\ I(f) &= \int_0^1 x^2\,dx = 1/3 \end{aligned}\right\} I(f) \neq \tilde{I}(f)$$

DoP of $\tilde{I}(f) = 1$

Ques: Determine c_1 & c_2 in the Quad. formula

$$I(f) = \int_{-1}^{1} f(x)\,dx \approx c_1 f(0) + c_2 f(1) = \tilde{I}(f)$$

so that $\tilde{I}(f)$ is exact for poly of degree 1.



example $\boxed{c_1 f(0) + c_2 f(1)}$ approx. for area under $f(x)$

sum of 2 areas

Back to Prob

$f(x) = 1 \rightarrow c_1 f(0) + c_2 f(1) = \int_{-1}^{1} 1\,dx$ $\hookrightarrow \int 1\,dx = x|^1$

Back to Prob

$f(x)=1 \longrightarrow c_1 f(0) + c_2 f(1) = \int_{-1}^{1} 1\,dx \quad \int_{-1}^{1} 1\,dx = x\Big|_{-1}^{1} = 2$

$\boxed{c_1 + c_2 = 2}$

$f(x) = x \longrightarrow c_1 f(0) + c_2 f(1) = \int_{-1}^{1} x\,dx \quad \frac{x^2}{2}\Big|_{x=-1}^{1}$

$c_1 * 0 + c_2 = 0$

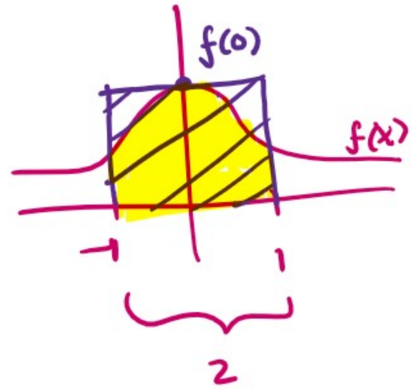
$$\boxed{\begin{aligned} c_1 + c_2 &= 2 \\ c_2 &= 0 \end{aligned}} \longrightarrow c_1 = 2 \quad c_2 = 0$$

formula becomes

$$\boxed{\hat{I}(f) = 2f(0)}$$

$I(f) = \int_{-1}^{1} f(x)\,dx$

Rectangle of width 2 & height $f(0)$.



integrate $f(x) = x$ bet. -1 & 1

$$\left.\begin{aligned} \int_{-1}^{1} x\,dx &= 0 \\ 2f(0) &= 2 * 0 = 0 \end{aligned}\right\}$$

Review of Exam 02:

* $\boxed{\text{Polynomial Interpolation}}$
  - Lagrange
  - Newton's D.D formula

(No formula for interpolatn will be provided)

Ques:

Construct a Lag. interpolating poly passing through:

$(-1, -1)$, $(0, 1)$, $(1, 2)$ 3 pieces of info

$x_0\ y_0 \qquad x_1\ y_1 \qquad x_2\ y_2$ $\Rightarrow$ deg. 2 poly $p_2^L(x)$

$p_2^L(x) = y_0 L_0(x) + y_1 L_1(x) + y_2 L_2(x)$

$= -L_0(x) + L_1(x) + 2L_2(x)$

$p_2(x) = y_0 \ldots$

$$= -L_0(x) + L_1(x) + 2L_2(x)$$

$$= -\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} + \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} + 2\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}$$

$$= -\frac{(x-0)(x-1)}{\underbrace{(-1-0)(-1-1)}_{\substack{-1 \quad -2 \\ 2}}} + \frac{(x+1)(x-1)}{\underbrace{(0+1)(0-1)}_{\substack{1 \quad -1 \\ -1}}} + \cancel{2}\frac{(x+1)(x-0)}{\underbrace{(1+1)(1-0)}_{\substack{\cancel{2} \quad 1 \\ 2}}}$$

$$= -\frac{x(x-1)}{2} - \frac{(x^2-1)}{1} + (x^2+x)$$

$$= \frac{-x^2+x}{2} - x^2+1 + x^2+x$$

$$\boxed{p_2^L(x) = -\frac{x^2}{2} + \frac{3}{2}x + 1}$$ Lagrange Poly

$p_2^L(-1) = -1/2 - 3/2 + 1 = -2+1 = -1$ ✓

$p_2^L(0) = 1$ ✓

$p_2^L(1) = -1/2 + 3/2 + 1 = 2$ ✓

<u>Ques</u>: Verify that $p_2^L(x)$ is same as poly obtained by Newton's D.D formula. $(-1,-1), (0,1), (1,2)$

$x_0\ y_0 \quad x_1\ y_1 \quad x_2\ y_2$

$y_0 \downarrow f(x_0) \quad y_1 \downarrow f(x_1) \quad y_2 \downarrow f(x_2)$

$$p_2^N(x) = f(x_0) + \overset{2}{f[x_0,x_1]}(x-x_0) + f[x_0,x_1,x_2](x-x_0)(x-x_1)$$

x_i	$f(x_i)$	$f[x_i,x_{i+1}]$
$\boxed{-1}$	-1	
		$\frac{1-(-1)}{0+1} = \boxed{\frac{2}{1}}$
$\boxed{0}$	1	
		$\frac{2-1}{1-0} = \boxed{1}$
1	2	

$$f[x_0,x_1,x_2] = \frac{f[x_1,x_2] - f[x_0,x_1]}{x_2-x_0}$$

$$\frac{1-2}{1-(-1)} = -\frac{1}{2}$$

$$p_2^N(x) = -1 + 2(x-(-1)) + (-1/2)(x-(-1))\,x$$

$$p_2^N(x) = -1 + 2(x-(-1)) + (-1/2)(x-(-1))\,x$$
$$= -1 + 2(x+1) + \left(-\tfrac{1}{2}\right)(x^2+x)$$
$$= -\frac{x^2}{2} + 2x - \frac{x}{2} + (-1+2)$$

$$\boxed{p_2^N(x) = -\frac{x^2}{2} + \frac{3}{2}x + 1} = \text{same as } p_2^L(x)$$

$$\{(-1,-1),\ (0,1),\ (1,2)\}$$

#2 (Review) Without computing the interpolating poly. estimate the error in interpolating $f(x) = \sin \pi x$ at $\boxed{x = 0, 0.5, 1, -1}$.

(No formula provided)

$$|f(x) - p(x)| \le ?$$

$\boxed{x = 0, 0.5, 1, -1}$ 4 points $\Rightarrow p_3(x)$

$$|f(x) - p_3(x)| \le ? \qquad -1 \le x \le 1$$

error Formula?

Relate it what we learnt

Taylor Remainder Formula:

$$|f(x) - p_n(x)| = \frac{(x-a)^{n+1}}{(n+1)!} f^{(n+1)}(c_x)$$

Taylor Poly

$f(x) = \sin \pi x$

Error Rep. for interpolating poly:

$$|f(x) - p_3(x)| = \left| \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{4!} f''''(c_x) \right|$$

($x_0 = 0$, $x_1 = 0.5$, $x_2 = 1$, $x_3 = -1$)

$$= \left| \frac{x(x-0.5)(x-1)(x+1)}{24} \pi^4 \underbrace{\sin \pi c_x}_{c_x = 0.5} \right| \qquad -1 \le x \le 1$$

$$= \left| x(x-0.5)(x-1)(x+1) \right| \pi^4 * 1$$

$$\le \left|\frac{x(x-0.5)(x-1)(x+1)}{24}\right| \pi^4 * 1$$

Bound $x(x-0.5)(x-1)(x+1)$ $\qquad -1 \le x \le 1$

do it individualy:

$-1 \qquad\qquad 1$

$|x| \le 1$

$|x-0.5| \le 1.5$ ($x = -1$)

$|x-1| \le 2$ ($x = -1$)

$|x+1| \le 2$ ($x = 1$)

$$|f(x) - p_3(x)| \le \left|\frac{x(x-0.5)(x-1)(x+1)}{24}\right| \pi^4$$

$$\le 1 * 1.5 * 2 * 2 \ \frac{\pi^4}{24}$$

$$= \frac{6}{24}\pi^4 = \frac{\pi^4}{4}$$

$$S(x) = \begin{cases} S_1 & 0 \le x \le 1 \\ S_2 & 1 < x < 2 \\ S_3 & 2 \le x \le 3 \end{cases}$$

Verify that given $s(x)$ is a spline

$\lim_{x \to 1^-} s''(x) = \lim_{x \to 1^+} s''(x)$ and s' & s,

$\lim_{x \to 2^-} s''(x) = \lim_{x \to 2^+} s''(x)$ and s' & s.

$\lim_{x \to 2^-} S''(x) = \qquad \qquad x \to 2^+$

Num Integration (formula provided)

$$f(x) = e^x \cos 4x$$

$$\int_{-\pi}^{\pi} f(x)\,dx \approx S_n(f)$$

$$| f(x) - S_4(f) | \leq \, ?$$ without calculating

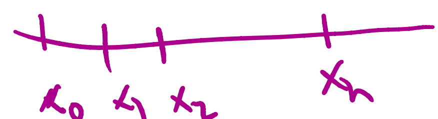
$S_4(f)$?

use formula: $\boxed{f(x) - S_4(f) = -\frac{(b-a)h^4}{180} f''''(c)}$

$$f(x) = e^x \cos 4x$$

How large should n be so that

$$\Big| \underbrace{\int_{-\pi}^{\pi} f(x)\,dx - T_n(f)}_{\text{error formula will be given.}} \Big| \leq 5 \times 10^{-6}$$



$$-\frac{(b-a)h^2}{12} f''(c)$$

$$h = \frac{b-a}{n}$$