Iterative Methods:

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$$x^{(0)} \longrightarrow x^{(1)} \dashrightarrow x^{*}$$

$$\boxed{Ax = b}$$

Last lecture: Jacobi method.

$$2x_1 - x_2 = 0 \rightarrow (1)$$
$$-x_1 + 2x_2 - x_3 = 1 \rightarrow (2)$$
$$-x_2 + 2x_3 = 2 \rightarrow (3)$$

Use Forward Elimination & Backward Substitution to solve the above system of linear equations.

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}\left[\begin{array}{ccc|c} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 1 \\ 0 & -1 & 2 & 2 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 2 & -1 & 0 & 0 \\ 0 & * & * & * \\ 0 & 0 & * & * \end{array}\right]$$

$$R_2 \rightarrow R_2 + 1/2\, R_1$$

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}\left[\begin{array}{ccc|c} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 1 \\ 0 & -1 & 2 & 2 \end{array}\right]$$

$$R_3 \rightarrow R_3 + 2/3\, R_2$$

$$\left[\begin{array}{ccc|c} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 1 \\ 0 & 0 & 4/3 & 8/3 \end{array}\right]$$

use Backward substitution

$$4/3\, x_3 = 8/3 \rightarrow \boxed{x_3 = 2}$$
$$3/2\, x_2 - x_3 = 1 \rightarrow$$
$$2x_1 - x_2 = 0$$

$$3/2\, x_2 - 2 = 1 \rightarrow \boxed{x_2 = 2}$$

$$2x_1 - x_2 = 0 \rightarrow \boxed{x_1 = 1}$$

$$\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \text{ solves } \boxed{\begin{array}{rl} 2x_1 - x_2 & = 0 \\ -x_1 + 2x_2 - x_3 & = 1 \\ -x_2 + 2x_3 & = 2. \end{array}}$$

Jacobi Method:

$$① \leftarrow 2x_1 - x_2 = 0$$
$$② \leftarrow -x_1 + 2x_2 - x_3 = 1$$
$$③ \leftarrow -x_2 + 2x_3 = 2$$

use eq^n ① $\longrightarrow x_1^{(1)}$
use eq^n ② $\longrightarrow x_2^{(1)}$
use eq^n ③ $\longrightarrow x_3^{(1)}$

based on inital $x^{(0)} = \begin{pmatrix} x_1^{(0)} \\ x_2^{(0)} \\ x_3^{(0)} \end{pmatrix}$

apply Jacobi method starting with

$$x^{(0)} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \longrightarrow x^{(1)} = \begin{pmatrix} 0 \\ 0.5 \\ 1 \end{pmatrix}$$

$$x_1^{(1)} \longrightarrow 2x_1^{(1)} - x_2^{(0)} = 0 \rightarrow x_1^{(1)} = 0/2 = 0$$

$$x_2^{(1)} \longrightarrow -x_1^{(0)} + 2x_2^{(1)} - x_3^{(0)} = 1 \rightarrow x_2^{(1)} = 1/2 = 0.5$$

$x_1^{(1)} \rightarrow \quad 2x_1^{(1)} - x_2^{(0)} = 0 \rightarrow x_1^{(1)} = 0/2 = 0$

$x_2^{(1)} \rightarrow \quad -x_1^{(0)} + 2x_2^{(1)} - x_3^{(0)} = 1 \rightarrow x_2^{(1)} = 1/2 = 0.5$

$x_3^{(1)} \rightarrow \quad -x_2^{(0)} + 2x_3^{(1)} = 2 \rightarrow x_3^{(1)} = 2/2 = 1$

$$x^* = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \quad x^{(1)} = \begin{pmatrix} 0 \\ 0.5 \\ 1 \end{pmatrix}$$

$$x^* - x^{(1)} = \begin{bmatrix} 1-0 \\ 1.5 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1.5 \\ 1 \end{bmatrix}$$

$\Rightarrow \|x^* - x^{(1)}\|$ = max of magnitude of each vector entry $= 1.5$

Based on $x^{(1)} = \begin{bmatrix} 0 \\ 0.5 \\ 1 \end{bmatrix}$ we calculate $x^{(2)}$ using Jacobi method

$$x^{(2)} = \begin{bmatrix} x_1^{(2)} \\ x_2^{(2)} \\ x_3^{(2)} \end{bmatrix} = \begin{bmatrix} 0.25 \\ 1 \\ 1.25 \end{bmatrix}$$

use eqⁿ ① for $x_1^{(2)}$:

$$2x_1^{(2)} - x_2^{(1)} = 0$$

$$2x_1^{(2)} = 0.5$$

$$\boxed{x_1^{(2)} = 0.25}$$

use eqⁿ ② for $x_2^{(2)}$:

$$-\underset{0}{x_1^{(1)}} + 2x_2^{(2)} - \underset{1}{x_3^{(1)}} = 1$$

$$2x_2^{(2)} - 1 = 1 \Rightarrow \boxed{x_2^{(2)} = 1}$$

use eqⁿ ③ for $x_3^{(2)}$:

$$-x_2^{(1)} + 2x_3^{(2)} = 2$$

$$2x_3^{(2)} = 2 + x_2^{(1)} = 2.5$$

$$\boxed{x_3^{(2)} = 1.25}$$

$$x^{(2)} = \begin{bmatrix} 0.25 \\ 1 \\ 1.25 \end{bmatrix} \rightarrow x^* - x^{(2)} = \begin{bmatrix} 0.75 \\ 1 \\ 0.75 \end{bmatrix}$$

$$\|x^* - x^{(2)}\| = 1.$$

More iterations will get us closer to the true solution.

In absence of x^*, we check the error by calculating $\|x^{(2)} - x^{(1)}\|$, $\|x^{(3)} - x^{(2)}\|$ . . .

Error Analysis for Jacobi Method

⊗ No info about exact soln x^*

* No info about exact soln x^*

* Derive upper bounds for error.

$$\| \underset{b}{Ax^*} - \underbrace{Ax^{(1)}} \| \leq ?$$

$\|Y\| \rightarrow$ norm of vector Y

$\|A\| \rightarrow$ maximum of row sum

$$\boxed{\|Ay\| \leq \underbrace{\|A\| \, \|y\|}}$$

Norm of product — product of norms.

Error Analysis for Jacobi Method

$$Ax = b$$

→ Predict (before making any calculations) if the Jacobi method converges or not?

We donot have a nice compact formula representing the Jacobi method.

e.g: Newton's method $f(x)=0$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

Before error analysis, we need to write a formula for Jacobi method:

Given $x^{(0)}$:

$$N^J x^{(1)} = b + P^J x^{(0)}$$

b → RHS vector $Ax=b$

N^J & P^J are matrices based on A which characterize Jacobi Method.

Jacobi Method

$$a_{11}x_1^{(1)} + a_{12}x_2^{(0)} + a_{13}x_3^{(0)} = b_1 \rightarrow (1)$$
$$a_{21}x_1^{(0)} + a_{22}x_2^{(1)} + a_{23}x_3^{(0)} = b_2 \rightarrow (2)$$
$$a_{31}x_1^{(0)} + a_{32}x_2^{(0)} + a_{33}x_3^{(1)} = b_3 \rightarrow (3)$$

$$a_{11}x_1^{(1)} = b_1 - a_{12}x_2^{(0)} - a_{13}x_3^{(0)}$$
$$a_{22}x_2^{(1)} = b_2 - a_{21}x_1^{(0)} - a_{23}x_3^{(0)}$$
$$a_{33}x_3^{(1)} = b_3 - a_{31}x_1^{(0)} - a_{32}x_2^{(0)}$$

Goal → $\boxed{N^J x^{(1)} = b + P^J x^{(0)}}$

$$x^{(1)} = \begin{pmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \end{pmatrix}$$

$$\begin{pmatrix} x_2^{(1)} \\ x_3^{(1)} \end{pmatrix}$$

$$N^J = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$

$$N^J x^{(1)} = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} \begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \end{bmatrix} = \begin{matrix} a_{11} x_1^{(1)} \\ a_{22} x_2^{(1)} \\ a_{33} x_3^{(1)} \end{matrix}$$

$$\boxed{N^J x^{(1)} = b + P^{(J)} x^{(0)}}$$

$$N^J = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}, \quad P^J = \begin{bmatrix} 0 & -a_{12} & -a_{13} \\ -a_{21} & 0 & -a_{23} \\ -a_{31} & -a_{32} & 0 \end{bmatrix}$$

$$= N^J - A$$

Written Jacobi method in a single Line:

$$N^J x^{(n+1)} = b + P^{(J)} x^{(n)}$$

where $N^J = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$

$$P^{(J)} = N^J - A.$$

error analysis for iterative methods $Ax = b$

Let $Ax^* = b$ i.e x^* is the true soln.

Given Jacobi iterative method:
for initial guess $x^{(0)}$,

$$N^J x^{(n+1)} = b + P^J x^{(n)} \qquad n = 1, 2, \ldots$$

Assume $x^{(n)} \to x^*$

$$\lim_{n \to \infty} x^{(n)} = x^*$$

$$N^J x^* = b + P^J x^*$$

$$- N^J x^{(n+1)} \qquad - N^J x^{(n+1)}$$

$$N^J x^* - N^J x^{(n+1)} = b + P^J x^* - \left(b + P^J x^{(n)}\right)$$

$$N^J \left(x^* - x^{(n+1)}\right) = P^J \left(x^* - x^{(n)}\right)$$

multiply by $(N^J)^{-1}$

multiply by $(N^J)^{-1}$ and use $(N^J)^{-1} N^J = \text{Identity matrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \ddots \end{bmatrix}$ we get:

$$x^* - x^{(n+1)} = (N^J)^{-1} P^J (x^* - x^{(n)})$$

$$\| x^* - x^{(n+1)} \| = \| (N^J)^{-1} P^J (x^* - x^{(n)}) \|$$

$n = 0, 1, 2, \dots$

$n = 0 \Rightarrow$

$$\| x^* - x^{(1)} \| = \| (N^J)^{-1} P^J (x^* - x^{(0)}) \|$$

error at 1st iteration $\longrightarrow$ error in initial guess $x^{(0)}$.

$n = 1$ $$\| x^* - x^{(2)} \| = \| (N^J)^{-1} P^J (x^* - x^{(1)}) \|$$

use: $\| \underbrace{Ax}_{(N^J)^{-1} P^J x} \| \leq \| \underbrace{A}_{\text{any matrix } (N^J)^{-1} P^J} \| \, \| x \|$ **for RHS**

$$\| x^* - x^{(2)} \| = \| (N^J)^{-1} P^J (x^* - x^{(1)}) \|$$

$$\leq \| (N^J)^{-1} P^J \| \, \underbrace{\| x^* - x^{(1)} \|}$$

$$= \| (N^J)^{-1} P^J \| \, \| (N^J)^{-1} P^J (x^* - x^{(0)}) \|$$

$$\leq \| (N^J)^{-1} P^J \| \, \| (N^J)^{-1} P^J \| \, \| x^* - x^{(0)} \|$$

$$= \| (N^J)^{-1} P^J \|^2 \, \| x^* - x^{(0)} \|$$

$$\Rightarrow \| x^* - x^{(n+1)} \| \leq \underbrace{\| (N^J)^{-1} P^J \|^{n+1}}_{(\text{Some\#})^{n+1}} \underbrace{\| x^* - x^{(0)} \|}_{\text{Initial error}}$$

Convergence of Jacobi is guaranteed

$$\boxed{\text{Convergence of Jacobi is guaranteed provided } \|(N^J)^{-1} P^J\| < 1}$$

for Jacobi method:

$$N^J x^{(n+1)} = b + P^J x^{(n)}$$

converges iff $\|(N^J)^{-1} P^J\| < 1$

Back to previous example:

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

$$N^J = \text{diagonal of } A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$P^J = N^J - A$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Check: $\|(N^J)^{-1} P^J\| < 1$?

$$(N^J)^{-1} = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.5 \end{bmatrix}$$

$$(N^J)^{-1} = \begin{bmatrix} \cdots \\ 0 & 0 & 0.5 \end{bmatrix}$$

$$(N^J)^{-1} P^J = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0 \\ 0 & 0 & 0.5 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$(N^J)^{-1} P^J = \begin{bmatrix} 0 & 0.5 & 0 \\ 0.5 & 0 & 0.5 \\ 0 & 0.5 & 0 \end{bmatrix} \begin{array}{l} \to 0+0.5+0 = 0.5 \\ \to 0.5+0+0.5 = 1 \\ \to 0.5 \end{array}$$

$$\|(N^J)^{-1} P^J\| = \max\{0.5, 1, 0.5\} = 1$$ convergence fails!

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

without calculating Jacobi iterates, remark on the convergence of the method!

$$\boxed{N x^{(n+1)} = b + P x^{(n)}}$$ convergence

$$\iff \|N^{-1} P\| < 1$$