Residual Correction Method

Tuesday, November 12, 2019 1:34 PM

Last Lecture: $0.001x_1 - x_2 = -1$

$x_1 + 2x_2 = 3$

Solving the above sys with 3 significant digits gave us the wrong solution.

$$\begin{bmatrix} 0.001 & -1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

Switched rows and performed G.E with 3 significant digits

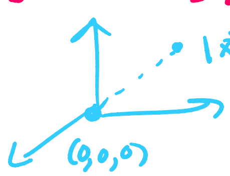
$$\frac{800}{801} \approx 0.9998\ldots$$

Residual Correction Iterative Method:

① Notion of ~~norm~~ magnitude of vector

$\vec{v} = \begin{bmatrix} -1 \\ 0 \\ -10 \end{bmatrix}$ $\|\vec{v}\| = \max\{1, 0, 10\}$
$= 10$

$\vec{x} = \begin{bmatrix} 2 \\ -1 \\ -0.5 \end{bmatrix}$ $\|\vec{x}\| = \max\{|2|, |-1|, |-0.5|\}$
$= 2$



② Notation $Ax^* = b$ $x^* \rightarrow$ True Solution to $Ax = b$.

Suppose $\hat{x}$ is an approximation to x^* obtained by solving $Ax = b$ with 4 digit precision.

Suppose x^* is known then

$\| x^* - \hat{x} \| \rightarrow$ magnitude of error.

norm of approxm error $x^* - \hat{x}$.

$$\underset{n \times n}{A} x = b \qquad n = 10^{10}$$

x^* might take a very long time to calculate.

Donot know the true value of x^*. x^* is not known,

trying to get a good approxm to it.

A & b we want to

trying to get a good approxm to it.

Just based on A & b, we want to make sure, we calculate $\hat{x}$ and the error $\hat{e} = x^* - \hat{x}$.

Algorithm: Residual Correction Method

(1) $x^{(0)} \rightarrow$ Solution to $Ax = b$ using finite precision.

(2) Compute: $r^{(0)} = b - Ax^{(0)}$ Residual

If $\|r^{(0)}\| < \varepsilon$ (user defined tolerance) then we accept $x^{(0)}$ as the solution to $Ax = b$.

else go to step 3.

(3) Correction Step: $x^{(1)} = x^{(0)} + e^{(0)}$ $\hookrightarrow x^* - x^{(0)}$

$e^{(0)}$ added to $x^{(0)}$

$e^{(0)}$ is calculated by solving:

$$A\boxed{e^{(0)}} = \underbrace{b - Ax^{(0)}}_{r^{(0)}} \text{ Residual}$$

Why this works?

If x^* was known, then

$$Ax^* = b - Ax^{(0)}$$

$$-Ax^{(0)}$$

$$A(\underbrace{x^* - x^{(0)}}_{e^{(0)}}) = b - Ax^{(0)} = r^{(0)}$$

(4) Update Step:

$$x^{(1)} = x^{(0)} + e^{(0)}.$$

(5) Replace $x^{(0)}$ with $x^{(1)}$ and go to (1).

$r^{(0)} = b - Ax^{(0)}$

Solve for $e^{(0)}$ as $A e^{(0)} = r^{(0)}$

solution correction to $x^{(0)}$.

$Ax = b \rightarrow$ G.Elimination (G.E)

save G.E information,

$Ax = b \rightarrow$ G.Elimination (G.E)

save G.E information,

then we can keeping

using it for solving

$Ae^{(0)} = r^{(0)}$.

See lecture slides.

Towards other Iterative Methods

① $\underset{n\times n}{A}x = b$

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$$

① $X^{(0)} \rightarrow$ initial Guess

② $X^{(1)}$ based on some formula.

$$x^{(0)} = \begin{bmatrix} x_1^{(0)} \\ x_2^{(0)} \\ \vdots \\ x_n^{(0)} \end{bmatrix} \rightarrow x^{(1)} = \begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \\ \vdots \\ x_n^{(1)} \end{bmatrix} \dots$$

③ Use norm of vector as a stopping criterea :

$\| x^{(1)} - x^{(0)} \| < \varepsilon$ stop else continue.

④ Matrix Norm:

$\downarrow$

magnitude

$$A = \begin{bmatrix} 1 & -1 & 0 \\ -10 & 2 & 5 \\ 6 & 0 & 500 \end{bmatrix} \begin{matrix} |1| + |-1| + 0 = 2 \\ 10+2+5 = 17 \\ 506 \end{matrix}$$

$\|A\|$ = max of Row sum of absolute values!

Norm can NEVER be negative!

$\|A\| = \max\{2, 17, 506\} = 506$

$\|A\|$ used in error analysis.

$\|Ax\|$ $\quad$ $\underbrace{A_{n\times n} \quad x_{n\times 1}}$

$\Downarrow$

Prop 1: $\|Ax\| \leq \|A\| \, \|x\|$

$\|A\|$ used also in deciding if an algorithm converges or not.

iterative Method

Iterative method 2: Jacobi Method

Consider foll. system:

$$
\begin{aligned}
① &\leftarrow 9x_1 + 9x_2 + 9x_3 = 1 \\
② &\leftarrow 2x_1 + 10x_2 + 3x_3 = 0 \\
③ &\leftarrow 3x_1 + 4x_2 + 11x_3 = 2
\end{aligned}
$$

Start with $x^{(0)}$ initial guess

$$x^{(0)} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$ typically the safest initial guess.

we want $x^{(1)} = ? = \begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \end{bmatrix}$ $\rightarrow$ using ① $1/9$, ② 0, ③ $2/11$

$① \Rightarrow 9x_1^{(1)} + 9x_2^{(0)} + 9x_3^{(0)} = 1 \Rightarrow 9x_1^{(1)} = 1$ or $x_1^{(1)} = 1/9$

update for $x_2^{(1)}$:

$② \quad 2x_1^{(0)} + 10x_2^{(1)} + 3x_3^{(0)} = 0 \Rightarrow x_2^{(1)} = 0$

$3x_1^{(0)} + 4x_2^{(0)} + 11x_3^{(1)} = 2 \Rightarrow x_3^{(1)} = 2/11$

Jacobi Method $\rightarrow$ method of Simultaneous Replacements

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General Form of Jacobi Method:

$$
\begin{cases}
a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\
a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \\
a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3
\end{cases}
$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

starting with $x^{(0)}$

$$x_1^{(1)} = \frac{1}{a_{11}}\left(b_1 - a_{12}x_2^{(0)} - a_{13}x_3^{(0)}\right)$$

$$x_2^{(1)} = \frac{1}{a_{22}}\left(b_2 - a_{21}x_1^{(0)} - a_{23}x_3^{(0)}\right)$$

$$x_3^{(1)} = \frac{1}{a_{33}}\left(b_3 - a_{31}x_1^{(0)} - a_{32}x_2^{(0)}\right)$$

$$X^{(1)} = D^{-1}b + D^{-1}Mx^{(0)}$$

$$D = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} \to D^{-1} = \begin{bmatrix} a_{11}^{-1} & 0 & 0 \\ 0 & a_{22}^{-1} & 0 \\ 0 & 0 & a_{33}^{-1} \end{bmatrix}$$

$$M = \begin{bmatrix} 0 & -a_{12} & -a_{13} \\ -a_{21} & 0 & -a_{23} \\ -a_{31} & -a_{32} & 0 \end{bmatrix}$$

$$M = D - A = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} - \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ \vdots & & \\ a_{31} & a_{32} & a_{32} \end{bmatrix}$$

verify $Mx^{(0)} =$