4-329-2

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$$Ax = b$$

WS10 :

$A_{n\times n}$ Gaussian Elimination is slow if n is large.
$O(n^3)$. $n=10^3$
operations very² large

we can reduce $O(n^3)$ to $2O(n^2)$ by LU Decomposition.

Gaussian Elimination: (Forward Elimination & Backward Substitution)

$$\begin{array}{lllll}
① & x_1 & + x_2 & + x_3 & = 1 \\
② & 2x_1 & + 4x_2 & + 4x_3 & = 2 \\
③ & 3x_1 & + 11x_2 & + 14x_3 & = 6
\end{array}$$

$$A \quad x \quad = b$$

convert to

upper triangular

$$\begin{bmatrix} * & * & * \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} * \\ * \\ * \end{bmatrix}$$

$$\begin{array}{l} R_1 \\ R_2 \\ R_3 \end{array}\left[\begin{array}{ccc|c} 1\, a_{11} & 1 & 1 & 1 \\ 2\, a_{21} & 4 & 4 & 2 \\ 3 & 11 & 14 & 6 \end{array}\right]$$

$$R_3 \left[\begin{array}{ccc|c} 3 & 11 & & 6 \end{array}\right]$$

$$R_2 \to R_2 - m_{21} R_1, \quad m_{21} = \frac{a_{21}}{a_{11}} = 2$$

$$R_3 \to R_3 - m_{31} R_1 \quad m_{31} = a_{31}/a_{11} = 3$$

m_{21} & m_{31} are called MULTIPLIERS.

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 0 \\ 3-3=0 & 11-3=8 & 14-3=11 & 6-3=3 \end{array}\right]$$

$8 \to 0$?

$$R_3 \to R_3 - \underset{4}{m_{32}} R_2, \quad m_{32} = \frac{a_{32}}{a_{22}} = 4$$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & 11-8=3 & 3-0=3 \end{array}\right]$$

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}}_{U} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

forward-elimination.

Backward Substitution:

$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ 2x_2 + 2x_3 &= 0 \\ 3x_3 &= 3 \end{aligned}$$

$x_3 = 1 \to \quad 2x_2 + 2x_3 = 0 \Rightarrow x_2 = -1$

$+ x_3 = 1$

$x_3 = 1 \rightarrow \quad 2x_2 + \ldots x_3 = \ldots$

$x_1 = 1$ using $x_1 + x_2 + x_3 = 1$

$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ as solution.

Check ✓ $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 4 \\ 3 & 11 & 14 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}$.

Gaussian Elimination $\rightarrow O(n^3)$
infeasible for large n.

$$A = LU$$

and instead solve the foll. 2 systems:

solve for y: $\boxed{Ly = b}$ $\begin{bmatrix} 1 & 0 & 0 \\ * & 1 & 0 \\ * & * & 1 \end{bmatrix} = L$

Solve for x: $Ux = y$.

$O(n^2)$ flops $\rightarrow$ reduce the complexity of solving $Ax = b$ for v. large n.

$Ax = b \xrightarrow[\text{Elim.}]{\text{forward}} Ux = \hat{b}$

$[A \mid b]$

$\downarrow$ Backward sub.

x

$A = LU \rightarrow \boxed{\begin{matrix} Ly = b_1 \\ Ux = y \end{matrix}}$

$\downarrow$

$$\boxed{\begin{aligned} L\hat{y} &= \hat{b} \\ Ux &= \hat{y} \end{aligned}}$$

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 4 \\ 3 & 11 & 14 \end{bmatrix}$$

$m_{21} = 2$

$m_{31} = 3 \qquad m_{32} = 4$

$$U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ m_{21} & 1 & 0 \\ m_{31} & m_{32} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix}$$

$$\boxed{A = LU}$$

Note: solving for $x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ amounts to:

solving $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ satisfying

$$Ly = b$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}$$

forward substitution

$y \qquad = 1$

forward substitution

$$\begin{aligned} y_1 &= 1 \\ 2y_1 + y_2 &= 2 \\ 3y_1 + 4y_2 + y_3 &= 6 \end{aligned}$$

$y_1 = 1$
$y_2 = 0$ $\rightarrow$ $\hat{b}$ after forward elimination
$y_3 = 3$

$$Ux = y$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

same as forward elimination!

$x_1 = 1$
$x_2 = -1$
$x_3 = 1$!

MATLAB: [L,U] = lu(A).

What happens when finite representation of numbers is taken into consideration?

Assume that you are using a calculator with 3 digits significand.

$$\frac{801}{800} \qquad 1/3$$

with 3 significant

Solve the system with 3 significant digits.

$$\begin{aligned} 0.001\,x_1 - x_2 &= -1 \\ x_1 + 2x_2 &= 3 \end{aligned}$$

$$\begin{array}{c} R_1 \\ R_2 \end{array} \left[\begin{array}{cc|c} 0.001 * 10^3 & -1 & -1 \\ 1 & 2 & 3 \end{array}\right]$$

$$R_2 \rightarrow R_2 - 10^3 R_1$$

$$\left[\begin{array}{cc|c} 0.001 & -1 & -1 \\ 1-1=0 & \underset{1002}{2+10^3} & \underset{1003}{3+10^3} \end{array}\right]$$

$$U \rightarrow \left[\begin{array}{cc|c} 0.001 & -1 & -1 \\ 0 & 1002 & 1003 \end{array}\right]$$

$$\begin{aligned} 0.001\,x_1 - x_2 &= -1 \\ 1002\,x_2 &= 1003 \end{aligned}$$

Backward Substitution:

$$x_2 = \frac{1003}{1002} = \underline{1.000998004} \approx 1$$

$x_2 = 1$ and $0.001x_1 - x_2 = -1$

$$0.001\,x_1 = -1 + x_2 = 0$$

$$x_1 = 0$$

$$x_1 = 0$$

$x_1 = 0$ and $x_2 = 1$ doesnot satisfy

$$x_1 + 2x_2 = 3.$$

This can be remedied by switching the equations : solve

$$x_1 + 2x_2 = 3$$
$$0.001x_1 - x_2 = -1$$

instead !

$$\left[\begin{array}{cc|c} 1 & 2 & 3 \\ 0.001 & -1 & -1 \end{array}\right] \longrightarrow \text{right solution}$$

$x_1 = 1$ and $x_2 = 1$

this satisfies: $x_1 + 2x_2 = 3$

and $0.001x_1 - x_2 = 0.001 - 1 = -1$

(Round off & 3 digits significance).

Worksheet 13: Solve

$$x_1 - \frac{800}{801}x_2 = 10,$$
$$-x_1 + x_2 = 50.$$

with $\frac{800}{801} = 0.9987515605.$

(a) $x_1 = 48010$, $x_2 = 48060$ solve the given system. (Easy to verify)

significant $800 = 0.99$[illegible]1

(b) Assume 2 significant digits. $\frac{800}{801} = 0.99 \approx 1$

$$
\begin{aligned}
① \leftarrow \quad & x_1 - x_2 = 10 \\
② \leftarrow \quad & -x_1 + x_2 = 50 \\
①+② \quad & 0 = 60
\end{aligned}
$$

inconsistent system!

(c) Assume 3 digits $\frac{800}{801}$ $\underbrace{0.998} \approx 1 = \frac{800}{801}$.

$$801 * \left(x_1 - \frac{800}{801} x_2 = 10 \right)$$

$$-x_1 + x_2 = 50$$

$$
\begin{aligned}
801 x_1 - 800 x_2 &= 8010 \\
-x_1 + x_2 &= 50
\end{aligned}
$$

solving the "scaled" system instead leads to the right solution!

Remark:

$$
\begin{aligned}
x_1 - \frac{801}{800} x_2 &= 10 \\
-x_1 + x_2 &= 50
\end{aligned}
$$

$\frac{801}{800} = 1.\underbrace{00125}$ poses a smaller risk

$\because$ admits exact repn.

$$\boxed{\underset{n\times n}{A} x = \underset{n\times 1}{b}}$$

$n \times n$ $\quad$ $n \times 1$

$n \approx 10^5$

<u>Direct methods</u> LU Decomp^n $O(n^2)$

<u>Indirect methods</u> take advantage of structure of A and solves the system in fewer than $O(n^2)$ operations.

→ Iterative Method : $f(x)=0$ (1D)

$x_0 \to \quad x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$

$\vdots$

$A = (a_{ij})_{n \times n}$

$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$ ← components of x

$x^{(0)} = \begin{pmatrix} x_1^{(0)} \\ x_2^{(0)} \\ \vdots \\ x_n^{(0)} \end{pmatrix}$ initial guess to $Ax=b$.

$x^{(1)} = \begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \\ \vdots \\ x_n^{(1)} \end{bmatrix} \cdots \longrightarrow A^{-1}b$

<u>Issue 1</u>: $\quad x^{(1)} = \begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \\ \vdots \\ x_n^{(1)} \end{bmatrix}$

how do I measure error ?

$x^* = A^{-1}b =$ vector $n \times 1$

$\|\cdot\| \quad \longrightarrow \|x^* - x^{(1)}\|$? vector norm (Magnitude

$\|\cdot\|$ $\longrightarrow \|x^* - x^{(1)}\|$? vector norm (Magnitude of vector)

Issue 2: measuring the quality of A.

Ans. to issue 1:

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\|x\|^2 = \sqrt{x_1^2 + x_2^2 + x_3^2}$$ (Norm/magnitude of x).

$$\|Ax\| = ?$$