Homework 4

Tuesday, November 5, 2019 1:29 PM

(a) $f(x) = e^x$ $x=0$

(b) $f(x) = \tan^{-1}(x^2-x+1)$, $x=1$

(c) $f(x) = \tan^{-1}(100x^2-199x+100)$ $x=1$

$h = 0.1, \frac{0.1}{2}, \frac{0.1}{2^2}, \frac{0.1}{2^3} \dots$

① $D_h^+ f$ ② $D_h^- f$ ③ $D_h f$ ⑨ $D_h^{(2)} f(x)$

$h = 0.5, \frac{0.5}{2}, \frac{0.5}{2^2}, \dots$

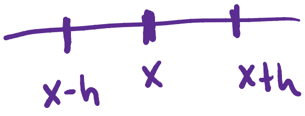
Method of undetermined coeffs.

$$\int_a^b f(x)\,dx \approx c_1 f(a) + c_2 f\left(\frac{a+b}{2}\right) + c_3 f(b) =: \hat{I}(f)$$

$f(x) = 1$

$f(x) = x$, $f(x) = x^2$.

$$\frac{d^2}{dx^2} f(x) \approx D_h^{(2)} f(x)$$



$$D_h^{(2)} f(x) = c_1 f(x-h) + c_2 f(x) + c_3 f(x+h)$$

Using Taylor expansions of f about x.

$$f(x+h) = f(x) + h f'(x) + \frac{h^2}{2} f''(x) + \frac{h^3}{3!} f'''(x).$$

$$f(\underset{x+h}{\underset{\downarrow}{x}}) = f(\underset{x}{\underset{\downarrow}{a}}) + \underbrace{(x-a)}_{h} f'(a) + \dots$$

$$f(x-h) = f(x) - h f'(x) + \frac{h^2}{2} f''(x) - \frac{h^3}{3!} f'''(x).$$

$$D_h^{(2)} f(x) = c_1 \Big(\overbrace{f(x) - h f'(x) + \frac{h^2}{2} f''(x) - \frac{h^3}{6} f'''(x)}^{f(x-h) \approx} \Big) +$$

$f(x-h)$

$$D_h^{(2)} f(x) = c_1\left(f(x) - h f'(x) + \frac{h^2}{2} f''(x) - \frac{h^3}{6} f'''(x)\right) +$$

$$c_2 f(x) +$$

$$c_3\left(f(x) + h f'(x) + \frac{h^2}{2} f''(x) + \frac{h^3}{6} f'''(x)\right)$$

$\uparrow f(x+h)$

$$D_h^{(2)} f(x) = (c_1 + \underset{0}{c_2} + c_3) f(x) + (-c_1 \underset{0}{+0} + c_3) h f'(x)$$

$$\underbrace{+ \underbrace{(c_1 + c_3)}_{1} \frac{h^2}{2} f''(x) + \underbrace{(-c_1 + c_3)}_{0} \frac{h^3}{6} f'''(x)}_{\text{num formula for } f''(x)}$$

$$f''(x) = D_h^{(2)} f(x)$$

$\Rightarrow$ coeff of $f(x) = 0 \Rightarrow c_1 + c_2 + c_3 = 0$ ①

$\Rightarrow$ coeff of $f'(x) = 0 \Rightarrow (-c_1 + c_3) h = 0$ ②

$\Rightarrow$ coeff of $f''(x) = 1 \Rightarrow (c_1 + c_3) \frac{h^2}{2} = 1$ ③

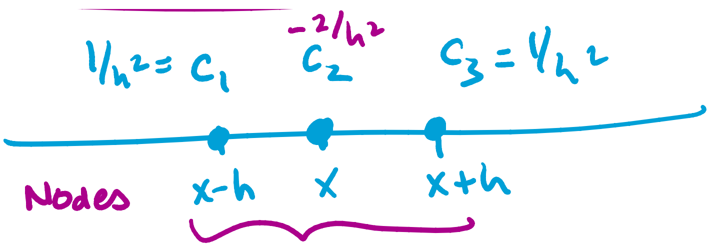
solve ② and ③ $\Rightarrow$ $\left.\begin{array}{l} c_1 = c_3 \\ c_1 + c_3 = 2/h^2 \end{array}\right\} \Rightarrow c_1 = c_3 = 1/h^2$

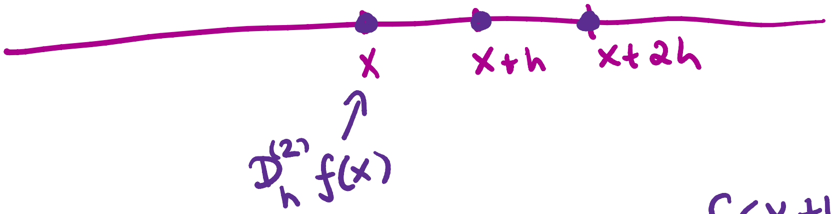
$$c_1 + c_2 + c_3 = 0 \Rightarrow 2/h^2 + c_2 = 0$$

$$\Rightarrow c_2 = -2/h^2$$

$$\boxed{\begin{aligned} f''(x) &\approx D_h^{(2)} f(x) \\ &= \frac{1}{h^2} f(x+h) - \frac{2}{h^2} f(x) + \frac{1}{h^2} f(x-h) \\ &= \left(f(x+h) - 2f(x) + f(x-h)\right)/h^2 \qquad (5.91) \end{aligned}}$$

$1/h^2 = c_1 \quad \overset{-2/h^2}{c_2} \quad c_3 = 1/h^2$





$$f''(x) \approx D_h^{(2)} f(x) = c_1 f(x) + c_2 f(x+h) + c_3 f(x+2h) \quad \rightarrow (5.93)$$

#10 (homework) wants to determine c_1, c_2, c_3

$$f''(x) \approx D_h^{(2)} f(x) \quad \rule{4cm}{0.4pt} \quad \rightarrow (5.94)$$

This method relies on Taylor expansion of f(x).
and can be applied to:

- integration
- differentiation
- interpolation

} method of undetermined coeffs.

$\boxed{\text{System of Linear Equations}}$

$$Ax = b \rightarrow [A \,\vdots\, b]$$

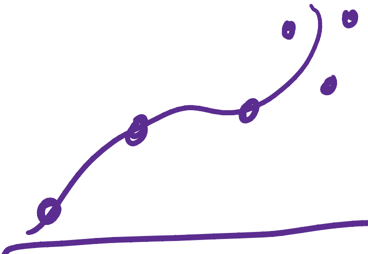
elementary Row operations

Review: Worksheet 10

→ Inverse*

→ matrix operations ...

$H_n \rightarrow$ HILBERT MATRIX. (Least Square approxm)



$$\boxed{H_n x = b}$$

$$H_n = (h_{ij})_{n \times n}$$

$$h_{ij} = \frac{1}{i+j-1} \qquad i = 1, 2, \ldots n \; ; \quad j = 1, 2, \ldots n$$

example: $n = 2$

$$H_2 = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 1/3 \end{bmatrix}.$$

Exam 02:

#1 → spline $\lim_{x \to 1^-} s'(x) \neq \lim_{x \to 1^+} s'(x)$

#2 → $f(x) = \cos \pi x$ $\qquad \underbrace{\underset{x_0}{x = 0}, \underset{x_1}{0.5}, \underset{x_2}{1}, \underset{x_3}{-1}}_{P_3(x)}$

(IV) 1

$$\overbrace{\qquad\qquad}^{}$$

$$P_3(x)$$

$$|f(x) - P_3(x)| = \left| \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)\, f^{(IV)}(c)}{4!} \right|$$

$$f^{(IV)}(c) \downarrow \quad \left| \frac{\pi^4 \cos \pi c}{24} \right|$$

$$= |\underset{\downarrow\, 1}{x}\,\underbrace{(x-0.5)}_{-1}\,\underbrace{(x-1)}_{-1}\,\underset{\downarrow\, 1}{(x+1)}|$$

$$(x^2 - 0.5x)(x^2-1)$$

$$\underbrace{\underbrace{(x^2 - 0.5x)}_{x=-1} \quad \underset{\downarrow\, x=0}{(x^2-1)}}_{1.5}$$

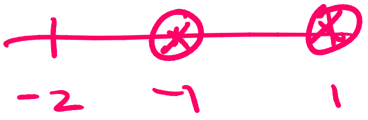
$$C=1$$

$$= |1*(-1-0.5)(-1-1)(1+1)| * \pi^4/24$$

$$(\underbrace{-1.5 * -2 * 2}_{6}) * \frac{\pi^4}{24}$$

$$= \frac{\pi^4}{4}$$

#3 $\displaystyle\int_{-2}^{1} f(x)\,dx \approx c_1 \underset{\downarrow\, -2}{f(-1)} + c_2 f(1)$



$$c_1 + c_2 = \int_{-2}^{1} 1\,dx = 3 \rightsquigarrow \text{wrong integration}$$

$$\boxed{-c_1} + c_2 = \int_{-2}^{1} x\,dx = -3/2$$

$$\swarrow -2c_1$$

$$\left.\begin{aligned} c_1 + c_2 &= 3 \\ -c_1 + c_2 &= -3/2 \end{aligned}\right\} \Rightarrow \begin{aligned} c_1 &= 9/4 \\ c_2 &= 3/4 \end{aligned}$$

#3(b) I

$$I = \int_{-1}^{1} \underset{\downarrow\, f(x)}{x^{10}}\,dx \approx f(-1/\sqrt{3}) + f(1/\sqrt{3})$$

$$= (-1/\sqrt{3})^{10} + (1/\sqrt{3})^{10}$$

3(b) II $I = \int_{-1}^{1} x^{10}\,dx = \frac{2}{11}$ $= 2/243$

$\{(1,3), (2,1), (3,2)\}$ → Lagrange poly

↙ Newton's D.D

$$p(x) = \frac{3}{2}x^2 - \frac{13}{2}x + 8$$

#5

$n=4$

$$I = \int_0^1 \boxed{xe^{-x^2}}\,dx$$

$$f(x) = \frac{x}{e^{x^2}} = xe^{-x^2}$$

$$= xe^{-(x**x)}$$

$$xe^{-x^2} \qquad xe^{(-x)^2} = xe^{x^2} \neq xe^{-x^2} = \frac{x}{e^{x^2}}$$

$h = 0.25$

$2*0.5e^{-0.5^2}$ (mistake)

$$T_4(f) = 0.125\left(f(0) + 2f(0.25) + \underbrace{2f(0.5}) + 2f(0.75) + f(1) \right)$$

$\simeq 0.8$

(b) $$\left| I - T_4(f) \right| = \left| -\frac{h^2}{12} f''(c) \right| \leq ?$$

$\qquad {}^{-c^2} \quad |$

$\downarrow$

$$= \frac{h^2}{12} \left| (4c^3-6c)e^{-c^2} \right|$$

$$\le \frac{h^2}{12} \underbrace{|4c^3-6c|}_{\downarrow c=1} \underbrace{|e^{-c^2}|}_{e^0}$$

$$= \frac{0.25^2}{12} * 2 * 1 \qquad = \frac{0.5 * 0.25}{12}$$

$\boxed{\text{Tues} / \text{Thurs}}$ Thanksgiving exam 03 makeup

Tues $\downarrow$ 26th Nov.

Matrix Algebra :

system of 2 eqns

$$c_1 x + c_2 y = b_1$$
$$c_3 x + c_4 y = b_2$$

$Ax = b$

$$\begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}^{-1} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

A = coefficient matrix

b = RHS vector (load vector)

3x3 matrix → (determinants)

$\downarrow$

3x3 Matrix

↓

① when does the inverse of A exist?

A^{-1} exists if $\det(A) \neq 0$

e.g: $A = \begin{bmatrix} c_1 & c_2 \\ c_3 & c_4 \end{bmatrix}$ det(A) = determinant of A

$= c_1 c_4 - c_2 c_3$

$\neq 0$

Construct poly interpolating f(x):

① <— $f(0) = y_1 \rightarrow a_0 = y_1$

② <— $f(1) = y_2 \rightarrow a_0 + a_1 + a_2 + a_3 = y_2$

③ <— $f'(0) = y_3 \rightarrow a_1 + 0 + \dots = y_3$

④ <— $f'(1) = y_4 \rightarrow a_1 + 2a_2 + 3a_3 = y_4$

poly: $p(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix}$$

Gaussian Elimination → Decomposition technique

$$Ax = b$$

Direct method → $x = A^{-1} b$ (closed form solution)

↳ poor $\det(A) \approx 0$

"Iterative Methods"

When $\det(A) \approx 0$,

Iterative Methods

When $\det(A) \approx 0$,

x_0

$$\left.\begin{array}{l} A x_1 \approx b \\ x_2 \rightarrow A x_2 \approx b \end{array}\right\}$$

$x_0, x_1, x_2, x_3, \ldots$

$x_0, x_1, x_2, \ldots \rightarrow A^{-1}b$