## Norm of a Matrix

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(6.87) textbook.

$$x = \begin{bmatrix} 1 \\ -5 \\ 0 \end{bmatrix}$$

$$\|x\| = 5 = \max\{1, 5, 0\}.$$

Distance Norm of a vector x i.e $\|x\|$

can never be negative!

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 2 & -5 & -2 \\ 0 & -0.5 & -100 \end{bmatrix} \begin{array}{l} \rightarrow \text{Row1} \rightarrow 2 \\ \rightarrow \text{Row2} \rightarrow 9 \\ \rightarrow \text{Row3} \rightarrow 100.5 \end{array}$$

$\|A\|$ = norm of a matrix A

"Distance of A from $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$"

Instead of max each entry $|a_{ij}|$

we do max of row sum.

$$\max\{\underbrace{1+1+0}_{2},\ \underbrace{2+5+2}_{4},\ \underbrace{0+0.5+100}_{100.5}\}$$

$$\|A\| = 100.5$$

Norm of a matrix used in error analysis of Iterative Methods.

Most useful property of $\|A\|$: $\hat{x}$ is approx. soln to $Ax = b$

$x^*$ be true solution to $Ax = b$. $x^*$ is unknown!

$$\| x^* - \hat{x} \| \leq \ldots \text{ in terms of } A, b, \hat{x}.$$

$$\boxed{\|Ax\| \leq \|A\| \, \|x\|}$$

$Ax$ = vector.

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Notation for Iterative Methods

## Notation for Iterative Methods

for root finding $f(x)=0$ (Scalar Equation)

Initial guess $x_0 \to$ $x_1 = x_0 - \dfrac{f(x_0)}{f'(x_0)}$

$|x_1 - x_0| < \varepsilon$ ?

if $|x_1 - x_0| > \varepsilon$

then $x_2 = x_1 - \dfrac{f(x_1)}{f'(x_1)}$

$\vdots$

$$\boxed{Ax = b}$$

$A = (a_{ij})_{n\times n}$ $\quad b = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$ $\quad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$

Typically $n = 3$ or $4$.

Initial Guess to solve $Ax = b$ : $x^{(0)} = \begin{pmatrix} x_1^{(0)} \\ x_2^{(0)} \\ \vdots \\ x_n^{(0)} \end{pmatrix}$

$x^{(1)}$ = Based on some formula

$x^{(1)} = \begin{pmatrix} x_1^{(1)} \\ \vdots \\ x_n^{(1)} \end{pmatrix}$.

**Jacobi Method :**

$$\begin{aligned}
&(1) \quad 2x_1 - x_2 &&= 0 \\
&(2) \quad -x_1 + 2x_2 - x_3 &&= 1 \\
&(3) \quad -x_2 + 2x_3 &&= 2
\end{aligned}$$

**Quick Exercise :** $\left.\begin{aligned} x_1 &= -1/2 \\ x_2 &= 1 \\ x_3 &= 3/2 \end{aligned}\right\} \to$ $2(-\frac{1}{2}) - 1 = -1-1 = -2$

Does not satisfy the first equation

$2x_1 = x_2 \Rightarrow 4x_1 = 2x_2$

in (2) $\quad -x_1 + 4x_1 - x_3 = 1 \Rightarrow \boxed{3x_1 - x_3 = 1}$

$2x_1 = x_2$ in (3) $\Rightarrow \boxed{-2x_1 + 2x_3 = 2}$

$$3x_1 - x_3 = 1$$
$$-2x_1 + 2x_3 = 2$$

$$\Rightarrow \quad 3x_1 - x_3 = 1$$
Divide by 2 $\Rightarrow$
$$-x_1 + x_3 = 1$$

$$2x_1 = 2 \Rightarrow x_1 = 1$$

$x_1 = 1$ & $x_2 = 2x_1$ gives $x_2 = 2$.

$x_2 = 2$ & $-x_2 + 2x_3 = 2$ gives $-2 + 2x_3 = 2$

$\Rightarrow x_3 = 2$.

$$\begin{array}{ccc} x_1 & x_2 & x_3 \end{array}$$
$$\left(5/3,\ 2,\ 2\right) \quad \alpha$$

$$2x_1 - x_2 = 0$$
$$2(5/3) - 2 \neq 0$$

$$\begin{array}{ccc} x_1 & x_2 & x_3 \end{array}$$
$$(0, 0, 1) \quad \alpha$$
$$2x_1 - x_2 = 0 \ \checkmark$$
$$-x_1 + 2x_2 - x_3 = 1$$
$$0 + 0 - 1 = 1 \ \times$$

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}\left[\begin{array}{ccc|c} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 1 \\ 0 & -1 & 2 & 2 \end{array}\right]$$

$$R_2 \to R_2 + \tfrac{1}{2} R_1$$

$$\begin{array}{c} \\ R_2 \\ R_3 \end{array}\left[\begin{array}{ccc|c} 2 & -1 & 0 & 0 \\ 0 & 2 - 1/2 = 3/2 & -1 & 1 \\ 0 & -1 & 2 & 2 \end{array}\right]$$

$$R_3 \to R_3 + \tfrac{2}{3} R_2$$

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array}\left[\begin{array}{ccc|c} 2 & -1 & 0 & 0 \\ 0 & 3/2 & -1 & 1 \\ 0 & 0 & 2 - \frac{2}{3} = \frac{4}{3} & 2 + \frac{2}{3} = 8/3 \end{array}\right]$$

(Forward ELIM.)

$R_3 \left[ 0 \quad 0 \quad 2 - \frac{2}{3} \left(\frac{1}{3}\right) \quad 1 + \frac{5}{3} \right]$ $= \frac{8}{3}$

Upper Triangular (Forward ELIM.)

$$
\begin{aligned}
2x_1 - x_2 \qquad &= 0 \\
\tfrac{3}{2}x_2 - x_3 &= 1 \\
\tfrac{4}{3}x_3 &= \tfrac{8}{3}
\end{aligned}
$$

Backward Subs.

$x_3 = 2$ from $\frac{4}{3}x_3 = \frac{8}{3}$

$\frac{3}{2}x_2 - x_3 = 1$ & using $x_3 = 2$ gives:

$\frac{3}{2}x_2 \quad = 1 + x_3 = 3$

$\boxed{x_2 = 2}$

use $x_3 = 2$, $x_2 = 2$ and $2x_1 - x_2 = 0$ to get

$2x_1 - 2 = 0 \Rightarrow x_1 = 1$

Solution: $\begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$ and check: ① ← $2x_1 - x_2 = 0$ ✓

② ← $-x_1 + 2x_2 - x_3$
$= -1 + 4 - 2$
$= 1$ ✓

③ → $-x_2 + 2x_3 = -2 + 4$
$= 2$ ✓

Try with
$$
\begin{aligned}
2x_1 - x_2 &= 2 \\
-x_1 + 2x_2 - x_3 &= 0 \\
-x_2 + 2x_3 &= 1.
\end{aligned}
$$

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Back to Problem

① ← $2x_1 - x_2 \qquad = 0$

② $-x_1 + 2x_2 - x_3 = 1$

③ $\qquad -x_2 + 2x_3 = 2$

we just saw $x_1 = 1$, $x_2 = x_3 = 2$ is the exact solution.

$x^{(0)} = \begin{pmatrix} 0 \\ \end{pmatrix}$ $x^{(1)} = \begin{pmatrix} x_1^{(1)} \\ \end{pmatrix}$

we just saw $x_1 = 1,\ x_2 = x_3 = 2$ is

solution.

Start with initial guess $X^{(0)} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \rightarrow X^{(1)} = \begin{pmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \end{pmatrix}$

Eqn ① used to calculate $x_1^{(1)}$

Eqn ② $\longrightarrow x_2^{(1)}$

Eqn ③ $\longrightarrow x_3^{(1)}$

$2x_1 - x_2 = 0 \longrightarrow 2x_1^{(1)} - x_2^{(0)} = 0$

$$\boxed{x_1^{(1)} = 0}$$

$x_2^{(1)}$ based on ②

$-x_1^{(0)} + 2x_2^{(1)} - x_3^{(0)} = 1 \rightarrow \boxed{x_2^{(1)} = 1/2}$

$x_3^{(1)}$ based on eqn ③:

$$-x_2^{(0)} + 2x_3^{(1)} = 2$$

$$\boxed{x_3^{(1)} = 1}$$

$X^{(0)} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

$\hookrightarrow X^{(1)} = \begin{bmatrix} 0 \\ 1/2 \\ 1 \end{bmatrix}$ Remember $x^* = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

$X^{(2)} = \ ? = \begin{bmatrix} x_1^{(2)} \\ x_2^{(2)} \\ x_3^{(2)} \end{bmatrix}$ based on $X^{(1)}$.

$x_1^{(2)} \rightarrow$ eqn ① $\quad 2x_1^{(2)} - x_2^{(1)} = 0$

$$x_1^{(2)} = 1/2$$

$x_2^{(2)} \rightarrow$ eqn ② $\quad -x_1^{(1)} + 2x_2^{(2)} - x_3^{(1)} = 1$

$\quad 0 \qquad 1$

$2x_2^{(2)} - 1 = 1 \Rightarrow \boxed{x_2^{(2)} = 1}$

$-x_2^{(1)} + 2x_3^{(2)} = 2$

$-0.5$

$X^{(0)} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $\quad -0.5 + 2x_3^{(2)} = 2 \Rightarrow 2x_3^{(2)} = 2.5$

$\Rightarrow x_3^{(2)} = 1.25$

$x^{(0)} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $\quad -0.5 + 2x_3^{(2)} = 2 \Rightarrow \ldots = 3$

$\Rightarrow \quad x_3^{(2)} = 1.25$

$x^{(1)} = \begin{pmatrix} 0 \\ .5 \\ 1 \end{pmatrix} \rightarrow x^{(2)} = \begin{bmatrix} 0.5 \\ 1 \\ 1.25 \end{bmatrix} \ldots \longrightarrow \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$

$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

← Row1 → diag Dominant (D.D)

Row2 → ''

Row 3 → D.D

Notice $|a_{11}| \geq |a_{12}|$ & $|a_{11}| \geq |a_{13}|$

$|a_{22}| \geq |a_{21}|$ & $|a_{22}| \geq |a_{23}|$

$|a_{33}| \geq |a_{31}|$ & $|a_{33}| \geq |a_{32}|$

→ Intelligent Initial Guess $x^{(0)} = \begin{pmatrix} b_1/a_{11} \\ b_2/a_{22} \\ b_3/a_{33} \end{pmatrix}$

$b_1 = 0 \qquad a_{11} = a_{22} = a_{33} = 2$

$b_2 = 1$

$b_3 = 2$

$$\left.\begin{aligned} x_1^{(0)} &= 0/2 = 0 \\ x_2^{(0)} &= 1/2 = 0.5 \\ x_3^{(0)} &= 2/2 = 1 \end{aligned}\right\}$$

exactly $x^{(1)}$ from previous calculation!

## Observations for Jacobi

① Method of simultaneous replacements

$$x^{(0)} \longrightarrow \begin{matrix} x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \end{matrix} \longrightarrow \begin{matrix} x_1^{(2)} \\ x_2^{(2)} \\ x_3^{(2)} \end{matrix}$$

$x_1^{(1)}$ $\qquad$ $x_2^{(1)}$ use "freshly alculated $x_1^{(1)}$"

"Gauss Seidel method".