Issues with Solving Systems

Tuesday, November 12, 2019 12:12 PM

Solve the foll. system with 3 significant digits.

$$0.\underline{00}1 x_1 - x_2 = -1 \qquad (1)$$

$$x_1 + 2x_2 = 3 \qquad (2)$$

multiply (2) with -0.001 and add to (1).

$$0.001x_1 - x_2 +$$
$$-0.001x_1 - 0.002x_2 = 3 * (-0.001) - 1$$

$$-1.002x_2 = -1.003$$

$$x_2 = \frac{-1.003}{-1.002} = 1.000998004$$

$$\simeq 1.000$$

(3 digits are allowed)

If x_2 is approximately 1 then from

(2) gives $x_1 = 3 - 2 = 1$

check: $0.001x_1 - x_2 = -1$ is satisfied

by $x_1 = x_2 = 1$. This is not true!

$\Rightarrow$ x_1, x_2 are not the solution for the system!

$$0.001x_1 - x_2 = -1$$
$$x_1 + 2x_2 = 3$$

Note: In numerical analysis,

$$Ax = b \text{ admits}$$

some error (due to finite representation) some error.(due to finite representation)

x^* is the <u>true</u> solution

$Ax \neq b \quad \hat{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$A\hat{x} = \hat{b}$

$e = x^* - \hat{x}$ error even when solving 2 equations due to 3 significant digits in arithmetic.

Can we avoid the error due to loss-of-significance?

$$0.001x_1 - x_2 = -1. \quad (1)$$

$$x_1 + 2x_2 = 3 \quad (2)$$

One Technique: Switch (1) and (2) and the solve the system.

$$(1)' \leftarrow x_1 + 2x_2 = 3$$

$$(2)' \leftarrow 0.001x_1 - x_2 = -1$$

multiply (2)' with $1/0.001 = 1000$ and subtract from (1)!

$$\begin{array}{rl} x_1 + 2x_2 & = 3 + 1000 \\ - x_1 + 1000x_2 & \end{array}$$

or even use one-step Gaussian elimination

$$1002x_2 = 1003$$

and we obtain the right solution satisfying (1)' and (2)'s

$$\begin{bmatrix} 1 & 2 \\ 0.001 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}$$

(right solution.

Right solution.

$Ax = b$

Numerically we might end up with:

$A\hat{x} = \hat{b}$

RESIDUAL CORRECTION METHOD:

Assumption ⊛ existing precision or digits of significance can be increased.

Let x^* be the exact solution to $Ax = b$.

suppose by using Gaussian Elim. with finite representⁿ we admit a solution $\hat{x}$, $\hat{x} \neq x^*$.

Introduce $e = x^* - \hat{x}$. (error)

$r = b - A\hat{x}$ (Residual)

under the assumptn of increased precision, we solve $\boxed{A\hat{e} = r}$

for true soln x^* $\quad Ax^* = b$

for approxm. $\hat{x}$ $\quad A\hat{x} = \hat{b}$

$$Ax^* - A\hat{x} = b - \hat{b}$$

$$A(x^* - \hat{x}) = r$$

upon solving $A\hat{e} = r$, we can add $\hat{e}$ to $\hat{x}$ then

$$\hat{x} + \hat{e}.$$

$$= \hat{x} + (x^* - \hat{x})$$

$$\hat{x} + \hat{e}$$
$$= \hat{x} + (x^* - \hat{x})$$
$$= x^* \quad (\underbrace{\text{True Solution}}!)$$

In theory the "equality" is exact but in numerical experiments,

$$\hat{x} + \hat{e} \simeq x^* \quad (\text{not } \underline{\text{exactly}}\ x^*)$$

Correction term is $\hat{e}$.

6.5.1 Residual Correction Method (Iterative Method)

I Let $x^{(0)} = \hat{x}$, where $\hat{x}$ is the solution to $Ax = b$ with some error (don't know the exact error $\because$ exact soln is unknown)

II Define Residual $r^{(0)} = b - Ax^{(0)}$ (measures the gap between $Ax^{(0)}$ and b)

III $\|r^{(0)}\| < \varepsilon \qquad \varepsilon \to$ user defined tolerance

magnitude of vector $r^{(0)}$

$$\boxed{\|r^{(0)}\| = \max_i |r_i^{(0)}|}$$

e.g: $\vec{v} = (1, -1, -2, 0) \qquad \|\vec{v}\| = \max\{|1|, |-1|, |-2|, 0\} = 2.$

If $\|r^{(0)}\| < \varepsilon$ the stop and declare $x^{(0)}$ as the solution.

else go to IV.

IV (Calculate the correction term)

Solve $\quad A e^{(0)} = r^{(0)}$

IV (Calculate the correction term)

Solve $A e^{(0)} = r^{(0)}$

Correction term.

using Gaussian Elimination or (LU-Decomp^n)

$[A \mid b] \to [U \mid b^*]$

save the row operations for $b \to b^*$

and use U.

V $x^{(1)} = x^{(0)} + e^{(0)}$

Gaussian Elim.

Correction term ($A e^{(0)} = b - A x^{(0)}$)

VI Replace $x^{(0)} \leftarrow x^{(1)}$ and go to step I.

Calculate
1. $r^{(0)} = b - A x^{(0)} \simeq 0.0000001$
2. $\|r^{(0)}\| < \varepsilon$
3. $\to$ solve for $e^{(0)}$ $A e^{(0)} = r^{(0)}$
4.
5. $x^{(1)} = x^{(0)} + e^{(0)}$

Look at lecture slides for an example.

Develop iterative methods to solve

$$\underset{3\times 3}{A}\, x = \underset{3\times 1}{b}.$$

Notation $x^{(0)}, x^{(1)}, x^{(2)}, x^{(3)}, \ldots$

$$X^{(0)} = \begin{pmatrix} x_1^{(0)} \\ x_2^{(0)} \\ x_3^{(0)} \end{pmatrix}$$

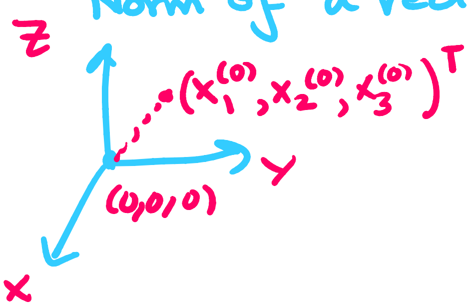
$$\boxed{Ax^{(0)} = b}$$

$$A = (a_{ij})_{3\times 3}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

NORM
Magnitude of Vector
Magnitude of a Matrix

Norm of a vector : $X^{(0)} = \begin{bmatrix} x_1^{(0)} \\ x_2^{(0)} \\ x_3^{(0)} \end{bmatrix}$



$$\|X^{(0)}\| = \max\{|x_1^{(0)}|, |x_2^{(0)}|, |x_3^{(0)}|\}.$$

Next Lecture: Norm of a matrix.