Matrix Algebra

Thursday, November 7, 2019 12:00 PM

$$Ax = b$$

W.sheet-10 Questions.

Refresh concepts esp. Gaussian Elimination

→ What goes wrong when we use finite representation

$$1/3 \approx 0.33$$

#5 WS10

(a) $A = 2\begin{bmatrix} 1 & 0 \\ -1 & 3 \end{bmatrix} + \begin{bmatrix} \boxed{1 \quad -1} \\ 1 \quad -1 \end{bmatrix} * \begin{bmatrix} 2 & 3 \\ 1 & 3 \end{bmatrix}$

Scalar mult   Matrix Mult.

$$= \begin{bmatrix} 2 & 0 \\ -2 & 6 \end{bmatrix} + \begin{bmatrix} \overset{1}{\boxed{2-1}} & \overset{0}{3-3} \\ \underset{1}{2-1} & \underset{0}{3-3} \end{bmatrix}$$

$$A = \begin{bmatrix} 3 & 0 \\ -2+1 & 6 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ -1 & 6 \end{bmatrix}$$

(b) $A^2 = \begin{bmatrix} 3 & 0 \\ -1 & 6 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ -1 & 6 \end{bmatrix}$

Do yourself!

③ $A = \begin{bmatrix} 5 & 6 & 1 \\ 2 & 2 & 6 \end{bmatrix}_{2\times 3}$ A^T = switch rows & coloumns

$$A^T = \begin{bmatrix} 5 & 2 \\ 6 & 2 \\ \ldots \end{bmatrix}$$

$$A' = \begin{bmatrix} 6 & 2 \\ 1 & 6 \end{bmatrix}_{3\times 2}$$

$$A * A^T = \begin{bmatrix} 62 & 38 \\ 28 & 44 \end{bmatrix}_{2\times 2}$$

coloumns = # rows

$A^T A$ = 3x3 matrix (do yourself!)

#4

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$A^{-1} = ?$

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right]$$

idea: carry out row (coloumn) operations so that

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/4 & -1/2 & 1/4 \\ 0 & 1 & 0 & -1/2 & 1 & -1/2 \\ 0 & 0 & 1 & 1/4 & -1/2 & 3/4 \end{array}\right]$$

verify these calculations!

A^{-1}

$$A * A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$$

$= A^{-1} * A.$

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Gaussian Elimination (& what can go wrong)

Solving a linear system:

$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ 2x_1 + 4x_2 + 4x_3 &= 2 \\ 3x_1 + 11x_2 + 14x_3 &= 6 \end{aligned}$$

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 4 \\ 3 & 11 & 14 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}$$

$$A x = b$$

Forward Elimination & Backward Substitution

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 0 & * & * & * \\ 0 & * & * & * \end{array}\right]$$

make zero

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 0 & * & * & * \\ 0 & 0 & * & * \end{array}\right]$$

eliminating non zero entries below the diag. entries a_{11}, a_{22}, a_{33}.

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & * & * \\ 0 & 0 & * \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ * \\ * \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & * \end{bmatrix} \begin{bmatrix} x_3 \end{bmatrix} \quad \begin{bmatrix} \ \end{bmatrix}$$

Backward Substitution

$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ * x_2 + * x_2 &= * \\ * x_3 &= * \end{aligned}$$

example:

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \left[\begin{array}{ccc:c} 1_{a_{11}} & 1 & 1 & 1 \\ 2_{a_{21}} & 4 & 4 & 2 \\ 3 & 11 & 14 & 6 \end{array}\right]$$

Step 1: 0

$$R_2 \to R_2 - \frac{a_{21}}{a_{11}} R_1$$

2 → multiplier m_{21}

$m_{31} = 3 \leftarrow R_3 \to R_3 - 3R_1$

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 2-2=0 & 4-2=2 & 4-2=2 & 2-2=0 \\ 3-3=0 & 11-3=8 & 14-3=11 & 6-3=3 \end{array}\right]$$

$$R_3 \to R_3 - \frac{8}{2} R_2 \quad 8/2 = m_{32}$$

upper triang. form

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 0 \\ 0 & 0 & \underset{3}{\overset{11-8}{\shortparallel}} & 3-0 \end{array}\right]$$

Backward Substitution

$$\begin{bmatrix} 1 & 1 & 1 \\ \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$$

$$\begin{array}{llr} (1) & x_1 + x_2 + x_3 & = 1 \\ (2) & \quad\;\; 2x_2 + 2x_3 & = 0 \\ (3) & \qquad\qquad 3x_3 & = 3 \end{array}$$

$(3) \Rightarrow \quad x_3 = 1$

$x_3 = 1 \;\&\; (2) \Rightarrow \quad 2x_2 + 2 = 0$

$\Rightarrow \quad x_2 = -1$

$x_2 = -1, x_3 = 1 \;\&\; (1) \Rightarrow \quad x_1 = 1$

check:

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 4 \\ 3 & 11 & 14 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 6 \end{bmatrix}$$

Popular method for n (ORDER size of matrix) being small.

Count total # operations for A $n \times n$ matrix : $O(n^3)$

If A is $10^3 \times 10^3$ matrix, then, the # operations is $\approx O(10^9)$ operations

Reducing the complexity of problem:

Reducing the [illegible]

problem:

$$A = LU$$

Lower Triangular matrix

Upper Triangular matrix

Solving: $Ax = b$ amounts to solving

2 subproblems: $\boxed{Ax = b} \rightarrow L\underbrace{Ux}_{y} = b$

① find y: $Ly = b \rightarrow O(n^2)$

② find x: $Ux = y \rightarrow O(n^2)$

LU-Decomposition method

how to find L & U?

$$Ax = b$$

$$[A \,\vdots\, b] \xrightarrow{\text{Row operations}} [U \,\vdots\, b^*]$$

$L = ?$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ m_{21} & 1 & 0 \\ m_{31} & m_{32} & 1 \end{bmatrix}$$

example: find the LU-decomposition of

$$A = \begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix}$$

find upper triangular matrix U

$$R_2 \rightarrow R_2 - m_{21}R_1, \quad m_{21} = \frac{a_{21}}{a_{11}} = 1$$

find upper ... U

$R_2 \to R_2 - m_{21}R_1$, $m_{21} = \frac{a_{21}}{a_{11}} = 1$

$R_3 \to R_3 - m_{31}R_1$, $m_{31} = \frac{a_{31}}{a_{11}} = 1$

$$\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 5 \\ 0 & 2 & 12 \end{bmatrix}$$

$2 \downarrow 0$

$R_3 \to R_3 - 2R_2$, $m_{32} = \frac{2}{1} = \frac{a_{32}}{a_{22}}$

$$U = \begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ m_{21} & 1 & 0 \\ m_{31} & m_{32} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$$

verify:

$$\overset{L}{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \end{bmatrix}} \overset{U}{\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & 5 \\ 0 & 0 & 2 \end{bmatrix}}$$

$$= \begin{bmatrix} 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix} = A$$

using $A = LU$, we can solve

$$\left.\begin{aligned} A\vec{x}_1 &= b_1 \\ A\vec{x}_2 &= b_2 \end{aligned}\right\} \text{ for diff. choices of RHS vector.}$$

$A\vec{x}_2 = b_2$ } choices of RHS vector.

Direct Method: Solving $Ax = b$ without any initial guess x_0 and sequence of $\{x_n\} \to A^{-1}b$.

$\boxed{\text{MATLAB}}$ → Matrix Laboratory (WS-11)

What goes wrong when finite representation calculations are involved? (WS-13)

$$x_1 - \frac{800}{801}x_2 = 10$$

$$-x_1 + x_2 = 50$$

$$\begin{bmatrix} 1 & -\frac{800}{801} \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 50 \end{bmatrix}$$

$$\frac{800}{801} = 0.9987515605\ldots$$

Suppose I am using a calculator with 2 digits after decimal and round-up ($0.99 \approx 1$)

$$\frac{800}{801} \approx 0.99 = 1$$

$$\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 10 \\ 50 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 \end{bmatrix}\begin{bmatrix} x_2 \end{bmatrix} = \begin{bmatrix} 50 \end{bmatrix}$$

$$\begin{cases} x_1 - x_2 = 10 \\ -x_1 + x_2 = 50 \end{cases}$$

→ solve for x_1 & x_2 No solution

$$0 = 60$$

No solution!

even if we increase the digits

$$\frac{800}{801} = 0.\underbrace{998}7515605$$

still encounter a problem.

$$801\left(*\; x_1 - \frac{800}{801}x_2 = 10\right)$$

$$-x_1 + x_2 = 50$$

$$801x_1 - 800x_2 = 8010$$

$$-x_1 + x_2 = 50$$

Solve for $\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{bmatrix} 48010 \\ 48060 \end{bmatrix}$.