Num Differentiation

Thursday, October 31, 2019 12:00 PM

Approximate $\int_0^3 f(x)\,dx$ using 3 POINT Gaussian Quadrature

formula

Recall: $\int_{-1}^{1} f(x)\,dx \simeq w_0 f(x_0) + w_1 f(x_1) + w_2 f(x_2)$

$x_0 = -0.7745$, $w_0 = 0.5555$

$x_1 = 0$ $\quad w_1 = 0.8888$

$x_2 = -x_0$ $\quad w_2 = 0.5555$

$$\int_0^3 f(x)\,dx \longrightarrow \int_{t=-1}^{1} f(t)\,J\,dt$$

J: Jacobian of transfm

$$0 \le x \le 3 \longrightarrow -1 \le t \le 1$$

$x = 0 \Rightarrow t = -1$
$x = 3 \Rightarrow t = 1$
} What is the formula for this transformation or "t" substitution?

Page 224, (5.61). (Do YOURSELF!)

Num. Integration → Quadrature Formula

$$I(f) = \int_a^b f(x)\,dx \longrightarrow w_0 f(x_0) + w_1 f(x_1) = \tilde{I}(f)$$

Exact for poly. of degree 1 : DoP = 1

example
$w_0 + w_1 = 1$
$w_0 = 1/2$

Formulas

$$\int_0^1 f(x)\,dx \simeq c_1 f(0) + c_2 f'(1/2) + c_3 f(1/2)$$

f' ↓ Derivative of $f(x)$

① Find c_1, c_2 & c_3 so that DoP is as large as possible.

3 unknowns c_1, c_2, c_3 ⇒ need 3 equations to solve

These 3 equations obtained by demanding:

$$c_1 f(0) + \underbrace{c_2 f'(1/2)} + c_3 f(1/2) = \int_0^1 f(x)\,dx$$

These 3 equations obtained by ...

$$c_1 f(0) + c_2 f'(1/2) + c_3 f(1/2) = \int_0^1 f(x)\,dx$$

for $f(x) = 1$

$f(x) = x$

and $f(x) = x^2$.

$f(x) = 1 \Rightarrow \quad c_1 + 0 + c_3 = \int_0^1 1\,dx = 1$

$f(x) = x \Rightarrow \quad 0 + c_2 + \frac{c_3}{2} = \int_0^1 x\,dx = 1/2$

$f(x) = x^2 \Rightarrow \quad 0 + c_2 * 2\left(\frac{1}{2}\right) + c_3 * \frac{1}{4} = \int_0^1 x^2\,dx = 1/3$

$f'(x) = 2x$

$$c_1 + c_3 = 1 \quad ①$$

$$c_2 + c_3/2 = 1/2 \quad ②$$

$$c_2 + c_3/4 = 1/3 \quad ③$$

②−③

$$c_2 + c_3/2 = 1/2 - 1/3$$

$$- c_2 - c_3/4$$

$$c_3/4 = 1/6 \Rightarrow \boxed{c_3 = 2/3}$$

from ② and using $c_3 = 2/3$ we have:

$$c_2 + 1/3 = 1/2$$

$$\Rightarrow \boxed{c_2 = 1/2 - 1/3 = 1/6}$$

using ① : $c_1 + c_3 = 1$

$$c_1 = 1 - 2/3 = 1/3$$

so, the integration formula which is exact for poly of degree 2 is given by:

$$\int_0^1 f(x)\,dx \approx \underbrace{1/3\, f(0) + 1/6\, f'(1/2) + 2/3\, f(1/2)}_{\tilde{I}(f)}$$

check if $\tilde{I}(f) \overset{?}{=} \int_0^1 f(x)\,dx$ for $f(x) = x^3$.

$f(x) = x^3 \rightarrow \boxed{3x^2}$

$$\tilde{I}(f) = \underset{f(0)}{1/3 * 0^3} + 1/6 * \underbrace{3 * (1/2)^2}_{f'(1/2)} + 2/3\, \underset{f(1/2)}{(1/2)^3}$$

$I(f) = \frac{1}{3}$ ...

$f(0)$ $\quad$ $f'(1/2)$ $\quad$ $f(1/2)$

$$= \frac{1}{2} * \frac{1}{4} + \frac{1}{12} = \frac{1}{4}\left(\frac{1}{2} + \frac{1}{3}\right)$$

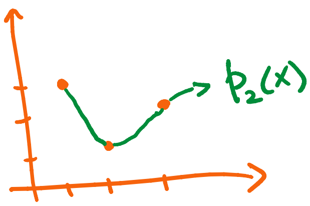
$$\hat{I}(f) = \frac{1}{4}\left(\frac{5}{6}\right) = 5/24$$

$$\int_0^1 x^3\,dx = 1/4 \implies I(f) \neq \hat{I}(f) \text{ so the}$$

DoP = 2

Numerical Differentiation

Given the foll. data:



x	$f(x)$
1	3
$\boxed{2}$	$\boxed{1}$
3	2

$\}\to f[1,2]$ Left Hand Approxm

$\}\ f[2,3] =$ Right Hand Approxm

Can we approximate $f'(2)$?

from Calculus:

$$\lim_{x\to 2^+} f(x) = \frac{f(3) - f(2)}{3-2} \approx f'(2) \quad \text{Right Hand Approxm}$$

or

$$\lim_{x\to 2^-} f'(x) = \frac{f(2) - f(1)}{2-1} \approx f'(2) \quad \text{left hand approxm}$$

$$\frac{dy}{dx} = F(x)$$

$$\frac{y(x+h) - y(x)}{h} = F(x) \quad \text{(Finite Difference Method)}$$

$D_h^+ f(x)$ = Right Hand Deri Approxm

$$= \frac{f(x+h) - f(x)}{h} = f[x, x+h]$$

$D_h^- f(x)$ = Left Hand Deri Approxm

$$= \frac{f(x) - f(x-h)}{h} = f[x-h, x]$$

... discrete data is given

h

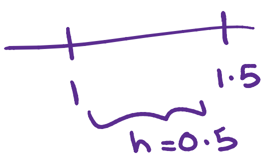
$D_h^+ f$ and $D_h^- f$ used when discrete data is given instead of $f(x)$.

If $f'(x)$ Does not exist at some x-value say a, we can always approximate the value numerically using $D_h^+ f$ & $D_h^- f$.

example $\quad f(x) = (x-1) \quad$ at $x=1$

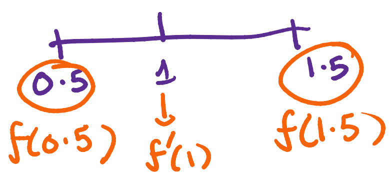
$f'(x)$ D.N.E

$$f'(1) \approx \frac{f(1+h) - f(1)}{h}$$



$$\approx f(1.5) - f(1)$$

$\hookrightarrow f(1)$ dne



$$\approx \frac{f(1.5) - f(0.5)}{1.5 - 0.5}$$

$\left.\begin{array}{l} D_h^+ f \\ D_h^- f \end{array}\right\}$ as $h \to 0$ $\left.\begin{array}{l} D_h^+ f(x) \\ D_h^- f(x) \end{array}\right\} \to f'(x)$

at a linear rate $O(h)$.

x	$f(x)$
1	3
2	1
3	2

$\to p_2(x) \approx f(x)$

$p_2'(x) \approx f'(x)$

$$p_2(x) = \frac{3x^2}{2} - \frac{13x}{2} + 8$$

$p_2'(2) \approx f'(2)$

$= 3x - 13/2$

$$P_2'(x) = 3 * \frac{2x}{2} - 13/2 = 3x - 13/2$$

$$\boxed{P_2'(2) = 6 - 13/2 = -1/2}$$

If I don't want to construct $P_2(x)$ to obtain

$P_2'(x) \simeq f'(x)$

then, formula is given by: (Page 235, (5.83))

$$D_h f(x) = \frac{f(\overset{x_2}{x+h}) - 2f(\overset{x_1}{x}) - f(\underset{x_0}{x-h})}{2h}$$ (Central Difference formula)

$x_0 = 1$ $\quad h = 1$

$x_1 = 2$

$x_2 = 3.$

<u>Note</u>: $D_h^+ f(x)$ and $D_h^- f(x)$ linearly converge to $f'(x)$

h	$D_h^+ f(x)$	$D_h^- f(x)$
1		
0.5		
0.25		
0.125		

observe linear convergence $O(h)$ order h

$D_h f(x)$ converges quadratically to $f'(x)$

h	$D_h f(x)$
1	0.5
0.5	0.25
0.25	⋮
0.125	

convergence $O(h^2)$

$D_h f$ is always preferred over $D_h^+ f$ and $D_h^- f$.

<u>Note</u>: The way we approximated $f'(x)$,

we can also approximate $f''(x)$.

we can also approximate $f''(x)$.

x_i	$f(x_i)$
x_0	$f(x_0)$
x_1	$f(x_1)$
x_2	$f(x_2)$
x_3	$f(x_3)$

$$f''(x_i) \approx D_h(D_h f(x_i))$$

example:

$$f''(x_1) \approx D_h\left(\frac{f(x_1+h) - 2f(x_1) - f(x_1-h)}{2h} \right)$$

Note: h = distance between x_{i+1} and x_i.

$$f''(x_1) \approx \frac{D_h(f(x_1+h)) - 2D_h f(x_1) - D_h f(x_1-h)}{2h}$$

Given the table below calculate

$D_{0.25} f(1)$ $\quad h=0.25$

$D_{0.5} f(1)$ $\quad h=0.5$

x_i	$f(x_i)$
0	1
0.25	2
0.5	4
0.75	6
1	7
1.25	-1
1.5	5
1.75	4
2	
2.25	
2.5	
2.75	
3	

0.25

$h=0.25$

$$D_{0.25} f(1) = \frac{f(1+h) - f(1-h)}{2h}$$

$$= \frac{f(1.25) - f(0.75)}{\underbrace{1.25 - 0.75}_{2h}}$$

$$D_{0.25} f(1) = \frac{-1 - 6}{0.5} = \frac{-7}{0.5}$$

$$= -14$$

$h = 0.5 \rightarrow 2h = 1$

$$D_{0.5} f(1) = \frac{f(1+0.5) - f(1-0.5)}{\underbrace{2h}_{1.5 - (0.5)}}$$

$$= \frac{5-4}{1}$$

$\boxed{D_{0.5} f(1) = 1}$ $\quad \boxed{D_{0.25} f(1) = -14}$.

$$\boxed{D_{0.5}f(1) = 1} \quad \boxed{D_{0.25}f(1) = -14}.$$

Application

(Year) Time	Population
0	100
10	200
20	300
30	500
40	1000
50	2000
60	8000

$$\frac{dy}{dt} = ky, \quad y(0) = 100$$

$y(t) \rightarrow$ population at time t.

(Parameter Estimation)

Next lecture: Sys of Linear Eq^n