Gauss Seidel

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Error Analysis: x^* is a solution to

$$Ax = b.$$

Then, $\|x^* - x^{(k+1)}\| \le \|N^{-1}P\|^{k+1} \|x^* - x^{(0)}\|$

where $N x^{(k+1)} = b + P x^{(k)}$ $k = 0, 1, \ldots$

for given choice of $x^{(0)}$.

$$\boxed{P = N - A}$$

For Jacobi,

$$N = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$$

$$P = N - A = \begin{bmatrix} 0 & -a_{12} & -a_{13} \\ -a_{21} & 0 & -a_{23} \\ -a_{13} & -a_{32} & 0 \end{bmatrix}$$

Any iterative method: given $x^{(0)}$

$$N x^{(k+1)} = b + P x^{(k)} \quad k = 0, 1, \ldots$$

converges if $\|N^{-1}P\| < 1$.

example : Given

$$2x_1 + x_2 = 1$$
$$x_1 + 2x_2 = 5$$

without calculating any iterates $x^{(1)}, x^{(2)}, \ldots$ based on $x^{(0)}$, check if Jacobi method converges.

$$Ax = b$$

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

Jacobi Method: Guess $x^{(0)}$

$$N^J x^{(k+1)} = b + P^J x^{(k)} \qquad k = 0, 1, \ldots$$

$$N^J = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

$$P^J = N^J - A$$ = All remaining entries of A with a neg. sign.

$$= \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

$$\|(N^J)^{-1} P^J\| < 1 \ ?$$

$$N^J = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \rightarrow (N^J)^{-1} = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}$$

$$(N^J)^{-1} P^J = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \underline{0} & -\underline{0.5} \\ -0.5 & 0 \end{bmatrix} \qquad \begin{matrix} 0 + 0.5 = 0.5 \\ 0.5 + 0 = 0.5 \end{matrix}$$

$$\|(N^J)^{-1} P^J\| = \max\{0.5, 0.5\} = 0.5 < 1$$

$\Rightarrow$ The Jacobi Method $N^J x^{(k+1)} = b + P^J x^{(k)}$ converges since $\|(N^J)^{-1} P^J\| < 1$.

$$\|x^* - x^{(k+1)}\| \leq \|(N^J)^{-1} P^J\|^{k+1} \underbrace{\|x^* - x^{(0)}\|}$$

$$\|x^* - x^{(k+1)}\| \leq \|(NJ)^{-1} PJ\| \, \underbrace{\|x^* - x^{(0)}\|}_{\text{error in initial guess}}$$

$$0 < \|(NJ)^{-1} PJ\|^{k+1} \quad \text{if } \|(NJ)^{-1} PJ\| < 1$$

$\longrightarrow 0$ as $k \to \infty$.

Example: #12 from 6.6

Consider the iterative method:

$$x^{(k+1)} = b + \alpha \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} x^{(k)}$$

$k = 0, 1, \ldots$

where α is an unknown number.

Find the values of α for which the above method converges. Irrespective of A, b & $x^{(0)}$.

$$N x^{(k+1)} = b + P x^{(k)}$$

where $N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $P = \alpha \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$.

Convergence of given iterative method:

$$x^{(k+1)} = b + \alpha \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} x^{(k)}$$

depends on $\|N^{-1} P\| < 1$ or not!

$$N^{-1} P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2\alpha & \alpha \\ \alpha & 2\alpha \end{bmatrix}$$

Row sum

$|2\alpha| + |\alpha|$

$\hookrightarrow \; N^{-1} \ldots \; \begin{bmatrix} \cdot & \cdot \\ 0 & 1 \end{bmatrix} \begin{bmatrix} \cdot & \cdot \\ \alpha & 2\alpha \end{bmatrix}$

Row sum

$$= \begin{bmatrix} 2\alpha & \alpha \\ \alpha & 2\alpha \end{bmatrix} \begin{array}{l} \to |2\alpha| + |\alpha| \\ \;\;\; |\alpha| + |2\alpha| \end{array}$$

For convergence, we need $\|N^{-1}P\| < 1$

$$\|N^{-1}P\| = \max\{ |2\alpha| + |\alpha|, \; |\alpha| + |2\alpha| \}$$
$$= 3|\alpha|$$

choose α such that

$$3|\alpha| < 1$$
$$|\alpha| < 1/3$$

$$-\frac{1}{3} < \alpha < 1/3.$$

Gauss Seidel Method (Method of SUCCESSIVE REPLACEMENTS)

$x_1^{(1)}$	$2x_1 - x_2$	$= 0$	$\to (1)$
$x_1^{(1)}, x_2^{(1)}$	$-x_1 + 2x_2 - x_3$	$= 1$	$\to (2)$
	$-x_1 + 2x_3$	$= 2$	$\to (3)$

formula for GS (Gauss Seidel)?

Diagonally Dominant matrix A $\Rightarrow$ GS method converge.

$$N^{GS} x^{(k+1)} = b + P^{GS} x^{(k)}$$

$$\|(N^{GS})^{-1} P^{GS}\| < 1 \quad \text{is}$$

equivalent to A being strictly diagonally dominant equivalent to A being strictly diagonally dominant.

Deriving N^{GS} & P^{GS} (remember $P^{GS} = N^{GS} - A$!)

Given $x^{(0)} = \begin{bmatrix} b_1/a_{11} \\ b_2/a_{22} \\ b_3/a_{33} \end{bmatrix}$ $b = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ $a_{11} = 2$, $a_{22} = 2$, $a_{33} = 2$.

$$x^{(0)} = \begin{bmatrix} 0 \\ 1/2 \\ 1 \end{bmatrix} \to \begin{bmatrix} 0 \\ 1 \\ 1.5 \end{bmatrix} \cdots \to \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

Derive GS $x^{(1)}$?

use $\boxed{2x_1 - x_2 = 0}$ to update $x_1^{(1)}$

$$2x_1^{(1)} - x_2^{(0)} = 0 \to x_1^{(1)} = 0$$

Use $x_1^{(1)} = 0$ and eq^n 2 to update $x_2^{(1)}$.

$$-x_1^{(1)} + 2x_2^{(1)} - x_3^{(0)} = 1$$

$$0 + 2x_2^{(1)} - 1 = 1$$

$$x_2^{(1)} = 1$$

Use $x_1^{(1)} = 0$ & $x_2^{(1)} = 1$ in eq^n (3) and update $x_3^{(1)}$

$$-x_2 + 2x_3 = 2$$

$$-x_2^{(1)} + 2x_3^{(1)} = 2$$

$$-1 + 2x_3^{(1)} = 2$$

$$-1 + 2x_3^{(1)} = 2$$

$$x_3^{(1)} = 3/2$$

What are N^{GS} and P^{GS} for any:

$$
\begin{aligned}
&① \quad a_{11}x_1^{(1)} + a_{12}x_2^{(0)} + a_{13}x_3^{(0)} = b_1 \\
&② \quad a_{21}x_1^{(1)} + a_{22}x_2^{(1)} + a_{23}x_3^{(0)} = b_2 \\
&③ \quad a_{31}x_1^{(1)} + a_{32}x_2^{(1)} + a_{33}x_3^{(1)} = b_3
\end{aligned}
$$

$Ax = b$

express ①-③ $\quad N^{GS} x^{(1)} = b + P^{GS} x^{(0)}$

$$\underbrace{\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{32} \end{bmatrix}}_{N^{GS}} \begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} + \begin{bmatrix} -a_{12}x_2^{(0)} - a_{13}x_3^{(0)} \\ -a_{23}x_3^{(0)} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} + \underbrace{\begin{bmatrix} 0 & -a_{12} & -a_{13} \\ 0 & 0 & -a_{23} \\ 0 & 0 & 0 \end{bmatrix}}_{P^{GS}} \begin{bmatrix} x_1^{(0)} \\ x_2^{(0)} \\ x_3^{(0)} \end{bmatrix}$$

$$N^{GS} x^{(k+1)} = b + P^{GS} x^{(k)}$$

where $\quad N^{GS} = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

where N ... $\begin{bmatrix} \cdots & \cdots & \cdots \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

$$P^{GS} = \begin{bmatrix} 0 & -a_{12} & -a_{13} \\ 0 & 0 & -a_{23} \\ 0 & 0 & 0 \end{bmatrix} = N^{GS} - A$$

$$\| (N^{GS})^{-1} P^{GS} \| < 1$$

For GS iterative method,

if A is strictly d.d i.e

$$|a_{12}| + |a_{13}| < |a_{11}|$$
$$|a_{21}| + |a_{23}| < |a_{22}|$$
$$|a_{31}| + |a_{32}| < |a_{33}|$$

$$\Rightarrow \quad \| (N^{GS})^{-1} P^{GS} \| < 1$$

example:

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

here A is strictly D.D $\Rightarrow$

GS method converges!

$\boxed{\text{LOOK LECTURE SLIDES}}$

LOOP

$$\boxed{Ax = b}$$

if $\det A \neq 0$
$\Rightarrow$ solution x exists
$A^{-1}b$.

Sometimes, the quality of A small changes in b cause large deviations in x.

example:

$$\underset{A}{\begin{bmatrix} 5 & 7 \\ 7 & 10 \end{bmatrix}} \underset{x}{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}} = \underset{b}{\begin{bmatrix} 0.7 \\ 1 \end{bmatrix}}$$

Replace b with $\begin{bmatrix} 0.69 \\ 1.01 \end{bmatrix} = \hat{b}$ Perturbed load vector/RHS vector

Solution to $A\hat{x} = \hat{b}$

is $\hat{x} = \begin{bmatrix} -0.17 \\ 0.22 \end{bmatrix}$

while solution to original problem: $Ax = b$

is $x = \begin{bmatrix} 0 \\ 0.1 \end{bmatrix}$

magnitude of perturbation $\|b - \hat{b}\| = 0.01$

magnitude of perturbation $\|b - \hat{b}\| = 0.01$

Ill Conditioned matrix : A is said to be ill conditioned if small changes in b

$b \rightarrow \hat{b}$ where $\|b - \hat{b}\| = \varepsilon \ll 1$ (much smaller than 1)

result in large deviations in the solutions to

$$Ax = b$$
$$A\hat{x} = \hat{b}.$$

Condition Number of a matrix :

$$\text{cond}(A, \text{inf}) = \|A\| \, \|A^{-1}\|$$

$$\|A * A^{-1}\| = \|I\| \quad I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$
$$\simeq 1$$

$$\|A\| \, \|A^{-1}\| \simeq 1$$

$\|A\| \, \|A^{-1}\| \gg 1$ then A is said to be ill conditioned.

$$A = \begin{array}{c} \begin{bmatrix} 5 & 7 \\ 7 & 10 \end{bmatrix} \end{array} \begin{array}{l} 12 \\ 17 \end{array} \qquad A^{-1} = \begin{bmatrix} 10 & -7 \\ -7 & 5 \end{bmatrix} \begin{array}{l} \text{ROW SUM} \\ \rightarrow 17 \\ \rightarrow 12 \end{array}$$

$\|A\| = \max\{12, 17\} = 17 \quad \|A^{-1}\| = 17$

$$\text{Cond}(A) = 17^2$$

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$\Rightarrow$ A is ill conditioned matrix & solutions will be sensitive to changes in b.

Homework 05 :

$$A_n = \begin{bmatrix} 1 & -1 & \cdots & -1 \\ & 1 & -1 & \cdots \\ & & \ddots & \\ & & & 1 \end{bmatrix}$$

input n $\longrightarrow$ returns A_n^{-1}

cond(An) $\longrightarrow$ cond(An, inf)

III

$$|\alpha| + |\beta| < 1$$

$$\|N^{-1}P\| = \|M\|.$$