Exam 03 Solutions

Tuesday, December 3, 2019 12:02 PM

I.

$$\begin{aligned} x_1 + x_2 + x_3 &= 1 \\ 2x_1 + 6x_2 + 8x_3 &= 3 \\ 6x_1 + 8x_2 + 18x_3 &= 5 \end{aligned}$$

1. Express the linear sys in the form $Ax = b$

$$\begin{bmatrix} 1 & 1 & 1 \\ 2 & 6 & 8 \\ 6 & 8 & 18 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

2. Use the LU factorization of the coeff matrix A to solve $Ax = b$.

$A = LU \longrightarrow \begin{matrix} Ly = b \\ Ux = y \end{matrix}$

$R_2 \to R_2 - 2R_1 \qquad m_{21} = 2$

$R_3 \to R_3 - 6R_1 \qquad m_{31} = 6$

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 6 \\ 0 & 2 & 12 \end{bmatrix}$$

$R_3 \to R_3 - m_{32} R_2 \,, \quad m_{32} = \dfrac{a_{32}}{a_{22}} = 1/2$

$$U = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 6 \\ 0 & 0 & 9 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 6 & 1/2 & 1 \end{bmatrix}$$

Mistake: $A = UL$

Forward Elim. & Backward substitution

$Ux = b$

forward

$$Ux = b$$

$$A = LU$$

Solve $\quad Ly = b$

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 6 & 1/2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

forward substitution

$$y_1 = 1$$

$$2y_1 + y_2 = 3 \rightarrow y_2 = 3 - 2 = 1$$

$$6y_1 + y_2/2 + y_3 = 5$$

$\boxed{Ly=b}$

$$6 + 1/2 + y_3 = 5$$

$$y_3 = 5 - 6.5 = -1.5$$

$$\underbrace{\begin{bmatrix} 1 & 1 & 1 \\ 0 & 4 & 6 \\ 0 & 0 & 9 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 1 \\ 1 \\ -1.5 \end{bmatrix}}_{y}$$

$$x_1 + x_2 + x_3 = 1$$

$$4x_2 + 6x_3 = 1$$

$$9x_3 = -1.5$$

$$9x_3 = -1.5 \Rightarrow x_3 = \frac{-1.5}{9} = -0.1667 \downarrow -1/6$$

$$4x_2 + 6(-0.1667) = 1$$

$$4x_2 - 1 = 1 \Rightarrow 4x_2 = 2$$

$$x_2 = 1/2$$

$$x_1 + x_2 + x_3 = 1 \Rightarrow x_1 + 1/2 - 1/6 = 1$$

$x_1 + x_2 + x_3 = 1 \Rightarrow x_1 + 1/2 - 1/6 = 1$

$x_1 + \frac{6-2}{12} = 1$

$x_1 + 4/12 = 1 \Rightarrow x_1 = 2/3$

Remark

$$\left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 2 & 6 & 8 & 3 \\ 6 & 8 & 18 & 5 \end{array}\right] \dashrightarrow \left[\begin{array}{ccc:c} 1 & 1 & 1 & 1 \\ 0 & 4 & 6 & 1 \\ 0 & 0 & 9 & -1.5 \end{array}\right]$$

y in $Ly = b$

II.

$$\begin{aligned} 3x_1 - x_2 &= -4 \\ 2x_1 + 5x_2 &= 2 \end{aligned}$$

Compute $x_J^{(k)}$ and $x_{GS}^{(k)}$ $k = 1, 2$ with the initial guess $x^{(0)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Do we have convergence? Why or why not?

Jacobi

$x^{(1)} = \begin{bmatrix} -4/3 \\ 2/5 \end{bmatrix}$ $\quad x^{(2)} = \begin{bmatrix} -18/15 \\ 14/15 \end{bmatrix}$

Gauss Seidel

$x^{(1)} = \begin{bmatrix} -4/3 \\ 14/15 \end{bmatrix}$ $\quad x^{(2)} = \begin{bmatrix} -46/45 \\ \frac{182}{225} \end{bmatrix}$

$3x_1 - x_2 = -4 \rightarrow$ 1 update x_1

$2x_1 + 5x_2 = 2 \rightarrow$ 2 update x_2

checking for convergence:

Diagonal Dominance of $\begin{bmatrix} 3 & -1 \\ 2 & 5 \end{bmatrix}$

implies convergence of GS method.

Jacobi method:

$$N x^{(k+1)} = b + P x^{(k)}$$

$$N = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix} \qquad P = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix} = N - A$$

$$N^{-1} = \begin{bmatrix} 1/3 & 0 \\ 0 & 1/5 \end{bmatrix} \qquad \|N^{-1}P\| < 1 \text{ (check!)}$$

Gauss Seidel

$$N = \begin{bmatrix} 3 & 0 \\ 2 & 5 \end{bmatrix} \qquad P = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$N^{-1} = \frac{1}{15-0}\begin{bmatrix} 5 & 0 \\ -2 & 3 \end{bmatrix} = \frac{1}{15}\begin{bmatrix} 5 & 0 \\ -2 & 3 \end{bmatrix}$$

$$1/15 \begin{bmatrix} 5 & 0 \\ -2 & 3 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$N^{-1}P = 1/15 \begin{bmatrix} 0 & 5 \\ 0 & -2 \end{bmatrix} \Rightarrow \|N^{-1}P\| = \max\{5/15, 2/15\} = 5/15 < 1$$

III. For the system $Ax = b$,

consider foll. iterative method/scheme

$$x_{n+1} = b + M x_n \qquad n = 0, 1, \ldots$$

Consider

$$x_{n+1} = b + Mx_n \qquad n = 0, 1, \ldots$$

$$M = \begin{bmatrix} \alpha & -0.5 \\ 0 & 2\alpha \end{bmatrix}$$ and α is unknown.

under what conditions on α will the iterative method converge?

$$\boxed{x_{n+1} = b + M x_n}$$

$$N x_{n+1} = b + P x_n$$ converges

provided $\|N^{-1}P\| < 1$

$$N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad P = M = \begin{bmatrix} \alpha & -0.5 \\ 0 & 2\alpha \end{bmatrix}$$

$\|N^{-1}P\| < 1$

$$N^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad N^{-1}P = M = \begin{bmatrix} \alpha & -0.5 \\ 0 & 2\alpha \end{bmatrix}$$

$\|M\| < 1$?

$$M = \begin{bmatrix} \alpha & -0.5 \\ 0 & 2\alpha \end{bmatrix} \Rightarrow \|M\| = \max\{|\alpha| + 0.5,\ 0 + |2\alpha|\}$$

Impose condn on α such that

$$\|M\| < 1$$

$$\max\{|\alpha| + 0.5,\ |2\alpha|\} < 1$$

means

$$|\alpha| + 0.5 < 1 \qquad \text{and} \qquad \frac{|2\alpha|}{2} < \frac{1}{2}$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$|\alpha| + 0.5 < 1 - 0.5$$

$$\Downarrow$$

$|\alpha| + 0.5 < 1 - 0.5$

-0.5

$$\Downarrow$$

$|\alpha| < 0.5$

$-$

$$\Downarrow$$

$|\alpha| < 1/2 = 0.5$

IV. Consider the interpolation problem of finding a quadratic poly. satisfying

$$p(1) = 2$$

$$p(-1) = 1 \quad \text{and} \quad p'(1/2) = 0.$$

Show that we can transform the above poly. interpolation problem to another problem of finding solution to

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & \underline{\underline{0}} \end{bmatrix} \begin{bmatrix} a^{a_2} \\ b^{a_1} \\ c^{a_0} \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

$p(x) = a_0 + a_1 x + a_2 x^2$

$p(1) = 2 \Rightarrow$ $\boxed{\begin{aligned} a_0 + a_1 + a_2 &= 2 \\ a_0 - a_1 + a_2 &= 1 \end{aligned}}$

$p(-1) = 1 \Rightarrow$

$p'(1/2) = 0 \rightarrow p'(x) = 0 + a_1 + 2a_2 x$

$p'(1/2) = a_1 + 2a_2(1/2) = a_1 + a_2$

$\boxed{a_1 + a_2 = 0}$

$\overset{c}{a_0} + a_1 + a_2 = 2$

$a_0 - a_1 + a_2 = 1$

$\boxed{\overset{b}{a_1} + \overset{a}{a_2} = 0}$

take

b a

V) 1. state whether true or false

$$x + y = 0$$
$$x + \frac{801}{800} y = 1$$

computer with 3 digits significance then the solution is $x = -800$ $y = 800$.

false

V 2.

$$5x + 7y = 0.7$$
$$7x + 10y = 1$$

is well conditioned FALSE

$$A = \begin{bmatrix} 5 & 7 \\ 7 & 10 \end{bmatrix} \longrightarrow \|A\| * \|A^{-1}\| = 17^2 >> 1$$