Gauss Seidel → Method of Successive Replacements

Tuesday, November 19, 2019 12:07 PM

Recall

$$2x_1 - x_2 = 0 \rightarrow (1)$$
$$-x_1 + 2x_2 - x_3 = 1 \rightarrow (2)$$
$$-x_2 + 2x_3 = 2 \rightarrow (3)$$

Initial Guess $x^{(0)} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \longrightarrow x^{(1)} = \begin{pmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \end{pmatrix}$

use eqⁿ 1 to update first component $x_1^{(1)}$

(same as Jacobi!)

$$2x_1^{(1)} - x_2^{(0)} = 0 \rightarrow x_1^{(1)} = 0.$$

use eqⁿ(2) to update the second component $x_2^{(1)}$.
and use freshly calculate $x_1^{(1)} = 0$

$$-x_1^{(1)} + 2x_2^{(1)} - x_3^{(0)} = 1$$

$$x_2^{(1)} = 1/2$$

use eqⁿ(3) to update third component $x_3^{(1)}$.
and use $x_1^{(1)} = 0$ and $x_2^{(1)} = 0.5$.

$$-x_2^{(1)} + 2x_3^{(1)} = 2$$

$$x_3^{(1)} = 1/2(2+0.5)$$
$$= 1.25$$

$$x^{(0)} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \longrightarrow x^{(1)}_{GS} = \begin{bmatrix} 0 \\ 0.5 \\ 1.25 \end{bmatrix} \ldots$$

$\begin{bmatrix} 0 \end{bmatrix}$ $\begin{bmatrix} 1.25 \end{bmatrix}$

See lecture slides for another example!

Error Analysis:

Pre Requisites

① $\underbrace{\|Ax\|}_{\text{norm of matrix vector product}} \leq \|A\| \, \|x\|$

"smaller than"

PRODUCT OF NORMS OF A AND x

② x^* true solution to $Ax = b$.

$\|x^* - x^{(k)}\| \leq$? without any knowledge about x^*!

③ Write the Jacobi & Gauss Seidel method in a compact form:

$$\boxed{N x^{(k+1)} = b + P x^{(k)}} \quad k = 0, 1, 2, \ldots$$

where N & P are matrices derived from A such that $P = N - A$.

Before error analysis, let us derive $N x^{(k+1)} = b + P x^{(k)}$ for Jacobi & GS (Gauss Seidel) Method.

Jacobi Method: $Ax = b$

Jacobi Method:

$$Ax = b$$

$$A = (a_{ij})_{3\times 3} \qquad x \,\&\, b \text{ are } 3\times 1.$$

Given $x^{(0)}$

$a_{11} \leftarrow \quad x_1^{(1)} = \dfrac{1}{a_{11}}\left(b_1 - a_{12}x_2^{(0)} - a_{13}x_3^{(0)}\right)$

$a_{22} \leftarrow \quad x_2^{(1)} = \dfrac{1}{a_{22}}\left(b_2 - a_{21}x_1^{(0)} - a_{23}x_3^{(0)}\right)$

$a_{33} \leftarrow \quad x_3^{(1)} = \dfrac{1}{a_{33}}\left(b_3 - a_{31}x_1^{(0)} - a_{32}x_2^{(0)}\right)$

$$N x^{(k+1)} = b + P x^{(k)} \qquad k=0$$

$$N x^{(1)} = b + P x^{(0)}$$

$$N = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix} \quad \text{check } N x^{(1)} = \begin{bmatrix} a_{11}x_1^{(1)} \\ a_{22}x_2^{(1)} \\ a_{33}x_3^{(1)} \end{bmatrix}$$

for Jacobi Method,

N = diag matrix with diag. entries a_{11}, a_{22}, a_{33}.

Then $\quad P = N - A$

$$P = \begin{bmatrix} 0 & -a_{12} & -a_{13} \\ -a_{21} & 0 & -a_{23} \\ -a_{31} & -a_{32} & 0 \end{bmatrix}.$$

What we have shown: Jacobi method is

$x^{(k+1)}$ ... $P\, x^{(k)} \qquad k=0,1,\dots$

What we have shown:

$$N x^{(k+1)} = b + P x^{(k)} \quad k = 0, 1, \ldots$$

given $x^{(0)}$

where $N = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$

$$P = N - A$$
$$= \begin{bmatrix} 0 & -a_{12} & -a_{13} \\ -a_{21} & 0 & -a_{23} \\ -a_{31} & -a_{32} & 0 \end{bmatrix}$$

Check: $N x^{(k+1)} = b + P x^{(k)}$

focus on RHS

$$b + P x^{(k)} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} + \begin{bmatrix} 0 & -a_{12} & -a_{13} \\ -a_{21} & 0 & -a_{23} \\ -a_{31} & -a_{32} & 0 \end{bmatrix} \begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \\ x_3^{(k)} \end{bmatrix}$$

$$b + P x^{(k)} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} + \begin{bmatrix} -a_{12} x_2^{(k)} - a_{13} x_3^{(k)} \\ -a_{21} x_1^{(k)} - a_{23} x_2^{(k)} \\ -a_{31} x_1^{(k)} - a_{32} x_2^{(k)} \end{bmatrix}}_{\text{RHS for Jacobi formula}}$$

RHS for Jacobi formula

$$Nx^{(k+1)} = \begin{bmatrix} a_{11} x_1^{(k+1)} \\ a_{22} x_2^{(k+1)} \\ a_{33} x_3^{(k+1)} \end{bmatrix}$$

For Gauss Seidel,

$$N x^{(k+1)} = b + P x^{(k)}$$

What are N & P ?

$$\boxed{P = N - A}$$

GS formula: Given $x^{(0)}$:

$$x_1^{(1)} = \frac{1}{a_{11}}\left(b_1 - a_{12} x_2^{(0)} - a_{13} x_3^{(0)}\right)$$

① $$\boxed{a_{11} x_1^{(1)} = b_1 - a_{12} x_2^{(0)} - a_{13} x_3^{(0)}}$$

$$\boxed{a_{21} x_1^{(1)} + a_{22} x_2^{(1)} = b_2 - a_{23} x_3^{(0)}}$$ ②

③ $$\boxed{a_{31} x_1^{(1)} + a_{32} x_2^{(1)} + a_{33} x_3^{(1)} = b_3}$$

$$Nx^{(k+1)} = b + Px^{(k)}$$

$$\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \\ x_3^{(1)} \end{bmatrix} \overset{\text{check}}{=} \begin{bmatrix} a_{11} x_1^{(1)} \\ a_{21} x_1^{(1)} + a_{22} x_2^{(1)} \\ a_{31} x_1^{(1)} + a_{32} x_2^{(1)} + a_{33} x_3^{(1)} \end{bmatrix}$$

$$\begin{bmatrix} a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x_3^{(1)} \end{bmatrix} \qquad \begin{bmatrix} a_{31}x_1^{(1)} + a_{32}x_2^{(1)} + \\ a_{33}x_3^{(1)} \end{bmatrix}$$

exactly the LHS of ①, ② & ③.

$$N = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$P = N - A$$
$$= \begin{bmatrix} 0 & -a_{12} & -a_{13} \\ 0 & 0 & -a_{23} \\ 0 & 0 & 0 \end{bmatrix}$$

What we have achieved:

GS formula:

$$N\, X^{(k+1)} = b + P x^{(k)} \quad k = 0, 1, \ldots$$

given $x^{(0)}$.

N = Lower triangular part of A including diag. entries

$$= \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$P = N - A = \begin{bmatrix} 0 & -a_{12} & -a_{13} \\ 0 & 0 & -a_{23} \\ 0 & 0 & 0 \end{bmatrix}.$$

Notation

N^{GS} & P^{GS} denote GS method

N^{J} & P^{J} " Jacobi method.

Lecture Slides.

Lecture Slides.

Coming to the Point: Error Analysis!

$$N x^{(k+1)} = b + P x^{(k)}$$

Error Analysis works for any iterative method than can be expressed using N & P,

example: III / HWK05 The error analysis is valid for:

we have $x^{(k+1)} = b + M x^{(k)}$

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with $M = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$

$$\boxed{N x^{(k+1)} = b + P x^{(k)}}$$

$$N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad P = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}.$$

Let $N x^{(k+1)} = b + P x^{(k)}$ denote an iterative formula for any $x^{(0)}$ for approximating x^* (x^* is the true solution to $Ax = b$).
(unknown)

Then, the iterative formula converges provided $\| N^{-1} P \| < 1$.

FURTHERMORE:

$$\| x^* - x^{(k+1)} \| \leq \| N^{-1} P \|^{k+1} \| x^* - x^{(0)} \|$$

$$\| x^* - x^{(k+1)} \| \leq \| N^{-1} P \| \; \| x - x \|$$

HWK05

$$A_n = \begin{bmatrix} 1 & -1 & \cdots & -1 \\ & & & \\ & & & \\ 0 & \cdots & 0 & 1 \end{bmatrix}$$

Given $n \longrightarrow A_n$

$A_n = \text{Identity}(n)$

for i

for j