Comparison Tests

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Warm up: ① $-\frac{2}{3} + \frac{4}{9} - \frac{8}{27} + \frac{16}{81} \cdots$

find the sum of the above series.

② Does the series $\sum_{n=0}^{\infty} 1/(2+3^n)$ converge?

③ Does the series $\sum_{n=0}^{\infty} (3/2)^n$ converge?

Answer: ① $\sum_{n=1}^{\infty} (-1)^n \frac{2^n}{3^n}$

$-\frac{2}{3} + 4/9 - 8/27 \cdots$

Geometric Series $ar + ar^2 + ar^3 + \cdots$

$a=1 \qquad r = -2/3$

$\frac{a}{1-r} =$

provided $|r| < 1 \qquad |r| = 2/3 < 1$

sum $\frac{1}{1-(-2/3)} = \frac{3}{5}$

Geom. Series $a + ar' + ar^2 + \cdots$

p-series $1 + 1/2^p + 1/3^p + \cdots \sum_{n=1}^{\infty} 1/n^p$

#2 $\sum_{n=0}^{\infty} 1/(2+3^n)$

$1/2 + 1/(2+3) + 1/(2+3^2) + \cdots$

Geom series $\sum^{\infty} 1/3^n$

looks close to Geom series $\sum_{n=0}^{\infty} 1/3^n$

converges since $r = 1/3 < 1$

Comparison Test $\quad 2 + 3^n > 3^n$

$$\underbrace{1/(2+3^n)}_{\text{Smaller Series}} < \underbrace{1/3^n}_{\text{Larger Convergent Series}}$$

$\sum_{n=0}^{\infty} 1/(3+2^n)$ converges.

$$\sum_{n=0}^{\infty} (3/2)^n \quad ?$$

$$(3/2)^0 + (3/2)^1 + (3/2)^2 + \ldots$$

$$r = 3/2 > 1$$

$\Rightarrow$ Geom Series $\sum (3/2)^n$ diverges

$\sum_{n=0}^{\infty} ar^n \quad$ if $|r| < 1 \Rightarrow \sum ar^n$ conv. with $\frac{a}{1-r}$

if $|r| > 1 \Rightarrow$ divergence

Quiz Ques: $\quad \frac{10}{2}, \frac{20}{3}, \frac{30}{4}, \frac{40}{5}, \ldots$

determine the general formula for above $\boxed{\text{sequence}}$ $\quad a_n = \frac{10n}{\quad} \quad n = 1, 2, 3, \ldots$

$\boxed{\text{sequence}}$ $a_n = \frac{10n}{n+1}$ $n = 1, 2, 3, \ldots$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{(10\cancel{n})/\cancel{n}}{(n+1)/n}$$

$$= \lim_{n\to\infty} \frac{10}{1 + 1/n}$$

$$\to 10$$

$$\sum_{n=0}^{\infty} a_n \qquad a_0 + a_1 + \ldots$$

$$\lim_{n\to\infty} a_n \stackrel{?}{=} 0$$

$$\sum_{n=0}^{\infty} (3/2)^n$$

$\lim_{n\to\infty} (3/2)^n \neq 0 \Rightarrow$ the series $\sum (3/2)^n$ diverges.

due to the divergence Test.

Comparison Test : $0 \leq a_n \leq b_n$

1. $\sum b_n$ converges $\Rightarrow \sum a_n$ converges
2. $\sum a_n$ diverges $\Rightarrow \sum b_n$ also diverges

Test $\sum^{\infty} 1/(n+1)$ for convergence.

Test $\sum_{n=1} 1/(n+1)$ for convergence.

$$1/2 + 1/3 + 1/4 + 1/5 + \dots$$

$\lim_{n\to\infty} 1/(n+1) = 0$ Divergence Test fails.

Comparison Test: $1/(n+1) < 1/n \quad n = 1, 2, \dots$

$n+1 > n$ $\sum_{n=1}^{\infty} 1/n^p$ $p=1$ Diverges!

$$\sum_{n=1}^{\infty} 1/n^p \longrightarrow \int_1^{\infty} 1/x^p \, dx = \lim_{b\to\infty} \int_1^b 1/x^p \, dx$$

$p=1$

$$= \lim_{b\to\infty} \int_1^b \frac{dx}{x}$$

$$= \lim_{b\to\infty} \ln x \Big|_1^b$$

$$= \lim_{b\to\infty} \ln b - \ln 1$$

$$= \infty$$

Integral Test directly to $\sum_{n=1}^{\infty} 1/(n+1)$

$$\sum_{n=1}^{\infty} 1/(n+1) = \int_1^{\infty} \frac{dx}{x+1}$$

$$= \lim_{b\to\infty} \ln(b+1) - \ln 2$$

$$= \lim_{b\to\infty} \ln(b+1) - \ln 2$$

$$= \infty$$

Divergence due to Integral Test!

$$\sum_{n=1}^{\infty} 1/(n+1)^3 \quad = \quad \int_1^{\infty} \frac{dx}{(x+1)^3}$$

$$= \lim_{b\to\infty} \int_1^{b} \frac{dx}{(x+1)^3}$$

$$= \lim_{b\to\infty} \left. \frac{(x+1)^{-3+1}}{-3+1} \right|_1^b$$

$$1/(n+1)^3 < 1/n^3 \quad n = 1, 2, \ldots$$

Bigger series Converges

$\Rightarrow$ Comparison Test smaller series $\Sigma\, 1/(n+1)^3$

also converges.

$$n < n+1 \Rightarrow 1/n > 1/n+1$$

$$n < n+1 \Rightarrow n^3 < (n+1)^3$$

$$\Rightarrow 1/n^3 > 1/(n+1)^3$$

$$\Rightarrow \quad 1/n^3 > 1/(n+1)^3$$

LIMIT COMPARISON TEST: Suppose $a_n > 0$

$b_n > 0$

and $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = L$, L is positive & finite.

Then $\displaystyle\sum_{n=0}^{\infty} a_n$ & $\displaystyle\sum_{n=0}^{\infty} b_n$ have the same behavior.

Application: $a_n = \dfrac{n^3+\sqrt{n}}{n^2+1}$ Apply Limit Comparison Test to determine the convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{n^3+\sqrt{n}}{n^2+1}$.

How to construct $b_n = ?$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L$$

$$\frac{a_n}{b_n} = \frac{n^3+\sqrt{n}}{n^2+1} \quad \frac{n^2}{n^3} \rightarrow 1/b_n$$

b_n is such that $1/b_n = \dfrac{n^2}{n^3} = 1/n$

$$\frac{a_n}{b_n} = \frac{n^3+\sqrt{n}}{n^2+1} * \frac{n^2}{n^3}$$

$$= \frac{(n^3+\sqrt{n})/n^3}{(n^2+1)/n^2} = \frac{\cancel{\frac{n^3}{n^3}} + \frac{\sqrt{n}}{n^3}}{\cancel{\frac{n^2}{n^2}} + 1/n^2}$$

$$= \frac{1 + n^{1/2-3}}{1 + n^{-2}}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1+n^{-2.5}}{1+n^{-2}} = 1$$

$\sum a_n$ & $\sum b_n$ have same behavior.

$\sum \frac{n^3+\sqrt{n}}{n^2+1}$ & $\sum b_n = \sum_{n=1}^{\infty} 1/n \to$ diverges

$\Rightarrow$ $\sum \frac{n^3+\sqrt{n}}{n^2+1}$ diverges.

$$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$$

$$\frac{a_n}{b_n} = \frac{\cancel{\sqrt{n}}}{n^2+1} \quad \frac{n^2}{\cancel{\sqrt{n}}}$$

$$\frac{1}{b_n} = \frac{n^2}{\sqrt{n}}$$

$$= \frac{1}{n^2+1} \quad n^2$$

b_n $\sqrt{n}$

n^2+1

$$= \frac{1}{\frac{n^2}{n^2} + 1/n^2} = \frac{1}{1 + 1/n^2}$$

$$\frac{a_n}{b_n} = \frac{1}{1+1/n^2} \longrightarrow 1 \text{ as } n \to \infty$$

$\sum a_n$ & $\sum b_n$ have the same behavior

$$b_n = \frac{\sqrt{n}}{n^2} = \frac{n^{0.5}}{n^2} = \frac{1}{n^{1.5}}$$

$\sum 1/n^{1.5}$ p series $p = 1.5 > 1$

$\Rightarrow$ $\sum a_n$ also converges.

example: $\sum 1/(n+1)$ ① Integral test Diverges

② Limit Comparison Test

$$\frac{a_n}{b_n} = \frac{1}{n+1} * \frac{n}{1} = \frac{n}{n+1}$$

$$= \frac{1}{1+1/n}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{1}{1+1/n} = 1$$

$\Rightarrow \sum a_n$ & $\sum b_n$ have same behavior.

$1/b_n = n \Rightarrow b_n = 1/n$ (Divergent series)

$\Rightarrow \sum 1/(n+1)$ diverges.

Alternating Series Test $a_n > 0 \quad n = 1, 2, \ldots$

$$\sum_{n=1}^{\infty} (-1)^n a_n = a_1 - a_2 + a_3 - a_4 \ldots$$

If $\lim_{n\to\infty} a_n = 0$

and $a_{n+1} < a_n$

$\Rightarrow \sum (-1)^n a_n$ converges.

example: $-\frac{2}{3} + 4/9 - 8/27 + \frac{16}{81} \ldots$

$n=1 \quad n=2 \quad n=3 \quad n=4$

$$\sum_{n=1}^{\infty} (-1)^n \frac{2^n}{3^n}$$

Alternating Series

$a_n = (2/3)^n$.

$\lim (2/3)^n = 0$

$$\lim_{n\to\infty} (2/3)^n = 0$$

$$a_{n+1} = (2/3)^{n+1} < (2/3)^n$$

$\Rightarrow \sum (-1)^n\, 2^n/3^n$ converges by Alternating series Test.

$$\sum_{n=1}^{\infty} \frac{\cos n\pi}{2^n} \to \frac{\overset{-1}{\cos\pi}}{2} + \frac{\overset{1}{\cos 2\pi}}{2^2} + \frac{\overset{-1}{\cos 3\pi}}{2^3} + \dots$$

Alternating series $\quad -1/2 + 1/2^2 - 1/2^3 + \dots$

$a_n = 1/2^n \qquad \lim_{n\to\infty} a_n = 0$ & $a_{n+1} < a_n$.

Note: $\sum_{n=1}^{\infty} 1/n \longrightarrow$ Diverges (p-series test $p=1$ diverges)

$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges. (using Alternating series Test)

$a_n = 1/n \qquad \lim_{n\to\infty} a_n = 0$ ✓

$a_{n+1} < a_n \qquad 1/(n+1) < 1/n$

$\Rightarrow \sum (-1)^n/n$ converges.