Sequences

Tuesday, November 12, 2019 2:58 PM

Saves time!

$$M_x = \rho \int_a^b \left(\frac{f(x)+g(x)}{2}\right)\left(f(x)-g(x)\right) dx$$

$$= \frac{\rho}{2} \int_a^b f(x)^2 - g(x)^2 \, dx$$

$\boxed{9.1}$

$a_1 \quad a_2 \quad a_3 \quad a_4$

$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \ldots$

$n=1 \quad n=2 \quad n=3 \quad \ldots$

$2^0 = 1$

$a_1, a_2, a_3 \ldots \qquad \{a_n\}$

$\frac{1}{2^0} \quad 1/2^1 \quad 1/2^2$

$$a_n = \frac{1}{2^{n-1}}$$

$$a_4 = \frac{1}{2^3} = \frac{1}{8}$$

Last Lecture: $a_1, a_2, a_3, \ldots \longrightarrow L$ convergence?

$$\lim_{n\to\infty} a_n = L$$

Replace a_n with $f(n)$

$$\lim_{n\to\infty} a_n = \lim_{x\to\infty} f(x)$$

$a_n = \frac{n-1}{}$

$$a_n = \frac{n-1}{2n+1}$$

find $\lim_{n\to\infty} a_n = ?$

$$a_n = f(x)$$

$$\lim_{n\to\infty} \frac{n-1}{2n+1} = \lim_{x\to\infty} \frac{x-1}{2x+1} \quad \frac{\infty}{\infty}$$

L'Hopital Rule

$$= \lim_{x\to\infty} \frac{1}{2} = \frac{1}{2}$$

Another approach to calculate

$\lim_{n\to\infty} \frac{n-1}{2n+1}$ (fraction term)

$$= \lim_{n\to\infty} \frac{\cancel{n}/\cancel{n} - 1/n}{2\cancel{n}/\cancel{n} + 1/n}$$

Divide num & deno with highest exponent of n that appears.

$$= \lim_{n\to\infty} \frac{1 - 1/n}{2 + 1/n} \to \frac{1-0}{2+0}$$

as $n \to \infty$
$1/n \to 1/\infty = 0$

example: $\lim_{n\to\infty} \frac{n-1}{3n^2+1} = ?$

Divide num & deno by n^2.

$$\lim_{n\to\infty} \frac{\cancel{n}/\cancel{n^2}_{n} - 1/n^2}{3\cancel{n^2}/\cancel{n^2} + 1/n^2}$$

$$\lim_{n\to\infty} \frac{\qquad}{\frac{3\cancel{n^2}}{\cancel{n^2}} + 1/n^2}$$

$$= \lim_{n\to\infty} \frac{1/n - 1/n^2}{3 + 1/n^2}$$

$$= \frac{0 - 0}{3 + 0} = 0$$

Note: $\lim_{n\to\infty} \frac{3n^2+1}{n-1}$ $\xrightarrow{\text{mult \& divide by } n^2}$

$$= \lim_{n\to\infty} \frac{3\cancel{n^2}/\cancel{n^2} + 1/n^2}{n/n^2 - 1/n^2} = \lim_{n\to\infty} n^2 \to \infty$$

$$\lim_{n\to\infty} \frac{n-1}{3n^2+1} = 0 \implies \lim_{n\to\infty} \frac{3n^2+1}{n-1} = \infty$$

$\frac{0}{\infty}$ indeterminant form

$$\lim_{n\to\infty} \frac{1}{6n} \to 0$$

$a_n \to 0$

$1/a_n \to \infty$

$$\lim_{n\to\infty} \frac{3n}{\sqrt{n^2-2}} \xrightarrow{\text{Divide by } n} \frac{3n/n}{(\sqrt{n^2-2})/n}$$

$$\lim_{n\to\infty} \frac{3}{\sqrt{\frac{\cancel{n^2}}{\cancel{n^2}} - \frac{2}{n^2}}}$$

$\to 3/$

$$= \lim_{n\to\infty} 3 \Big/ \sqrt{1 - 2/n^2} \to 3/\sqrt{1-0}$$

$$= 3$$

<u>example</u>: Recall $n!$ = product of first n natural numbers.

↳ factorial of n

$0! = 1$

$1! = 1$

$2! = 2*1 = 2$

$3! = 3*2*1 = 6$

$4! = 4*3*2*1 = 24$

$4! = 4*3! \Rightarrow \frac{4!}{3!} = 4$

⋮

$$\frac{(n+1)!}{n!} = \frac{(n+1)*n!}{n!} = n+1.$$

$a_n = \frac{n!}{(n+1)!}$ Determine $\lim_{n\to\infty} a_n = ?$

$$a_n = \frac{\cancel{n!}}{(n+1)*\cancel{n!}} = \frac{1}{n+1} \longrightarrow 0 \text{ as } n\to\infty.$$

<u>Note</u>: $2n! = 2*n!$ or $\underbrace{(2n)!}$ factorial of $2n$

Note: $2n! = 2 * n!$ or $(2n)!$

2 * factorial of n

$2 * n!$	$(2n)!$
$2*1$	$2!$
$2*(1*2)$	$4!$
$\vdots$	$6!$

20/ Textbook: $\dfrac{(4n+1)!}{(4n)!}$ Simplify the expression

$n = 0$ or 1 or $2, \ldots$

$$(4n+1)! = (4n+1) * 4n * (4n+1-1) \ldots$$

$$\frac{(4n+1)!}{(4n)!} = \frac{(4n+1) * \cancel{(4n)!}}{\cancel{(4n)!}} = 4n+1$$

Squeeze Theorem (Predict convergence of a_n)

$$a_n = \frac{\sin n\pi}{n}$$

$$\begin{array}{ccccc} c_n & \leq & a_n & \leq & b_n \\ \downarrow & & \downarrow & & \downarrow \\ L & & L & & L \end{array}$$

If $c_n \to L$ and $b_n \to L$ and

$$c_n \leq a_n \leq b_n$$

then $\underline{a_n \to L}$

then $a_n \to L$

$$\boxed{a_n = \frac{\sin n\pi/2}{n}}$$

$\sin n\pi$ $\quad n \to$ natural number $\quad \sin n\pi = 0$

$\sin n\pi/2$

$$a_1 = \frac{\sin \pi/2}{1} = \frac{1}{1} = 1$$

$$a_2 = \frac{\sin(2\pi/2)}{2} = 0$$

$$a_3 = \frac{\sin(3\pi/2)}{3} = \frac{-1}{3}$$

$\vdots$

$$-1 \leq \sin x \leq 1$$

$$\Rightarrow \underset{c_n}{\frac{-1}{n}} \leq \frac{\sin(n\pi/2)}{n} \leq \frac{1}{n} \to b_n$$

$$c_n = \frac{-1}{n} \leq \underset{a_n}{\frac{\sin(n\pi/2)}{n}} \leq 1/n = b_n$$

$$\frac{-1}{n} \to 0 \qquad 1/n \to 0$$

$\Rightarrow \quad a_n \to 0 \quad$ as $\quad n \to \infty$

$\Rightarrow \quad a_n \to 0 \quad$ as $\quad n \to \infty$

Because of squeeze Theorem.

Theorem: If a sequence $\{a_n\}$ satisfies the following properties then, it converges.

Property 1: $\{a_n\}$ is bounded

$$|a_n| \leq K$$

example: $a_n = \sin(n\pi/2) \Rightarrow |a_n| \leq 1$

$\{\sin(n\pi/2)\}$ is bounded.

Property 2: $\{a_n\}$ is monotonic

either

$$a_1 \geq a_2 \geq a_3 \geq a_4 \ldots \text{(Decreasing)}$$

OR

$$a_1 \leq a_2 \leq a_3 \leq a_4 \ldots \text{(Increasing)}$$

example: $1, 1/2, 1/4, 1/8, 1/16, \ldots$

$a_1 \geq a_2 \geq a_3 \ldots$ Decreasing

① $a_n = 1/2^n \qquad |a_n| \leq 1$

$\{a_n\}$ is Decreasing

$\Rightarrow \{a_n\}$ converges

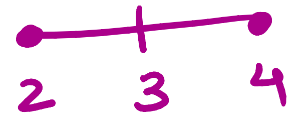
② $a_n = 3 + (-1)^n$

check Bounded? $\quad |a_n| \leq ?$

$a_1 = 3 + (-1)^1 = 3 - 1 = 2$

$a_1 = 3+(-1)^1 = 3-1 = 2$

$a_2 = 3+(-1)^2 = 3+1 = 4$



$a_3 = 2$, $a_4 = 4, \dots$

odd subscripts $\longrightarrow$ $a_1 = 2$, $a_3 = 2$, $a_5 = 2$

even subscripts $\longrightarrow$ $\left.\begin{matrix} a_2 \\ a_4 \\ a_6 \end{matrix}\right\} \longrightarrow 4$

$$|a_n| \le 4$$

$$\boxed{a_n = 3 + (-1)^n}$$

not convergent because

$$a_n = \begin{cases} 2 & \text{if } n = 1, 3, 5, 7 \dots \\ 4 & \text{if } n = 2, 4, 6, \dots \end{cases}$$

Theorem: $|a_n| \le K$ Bounded

and $\{a_n\}$ $\uparrow$ or $\downarrow$ seq

$$\Downarrow$$

$\{a_n\}$ converges.

$a_n = \dfrac{2n}{\dots}$ $\longrightarrow$ $\lim\limits_{n \to \infty} a_n = 2$

$$a_n = \frac{2n}{n+1} \rightarrow \lim_{n\to\infty} a_n = 2$$

$|a_n| \leq 2$ and a_n is increasing or decreasing ?

$$a_1 = \frac{2}{3} \quad a_2 = \frac{4}{3} \quad a_3 = \frac{6}{4} \ldots$$

$$a_n \overset{?}{\leq} a_{n+1}$$

$$\frac{2n}{n+1} \qquad \frac{2(n+1)}{(n+1)+1}$$

$$n+1 < (n+1)+1 = n+2$$

$$\boxed{(n+1)^{-1} > (n+2)^{-1}}$$

$$\frac{1}{n+1} > \frac{1}{n+2}$$

$$2n < 2n+2$$

$$f(x) = \frac{2x}{x+1} \rightarrow f'(x) > 0 \quad f(x) \text{ is increasing}$$

$$f'(x) < 0 \quad f(x) \downarrow$$

$$f'(x) = \frac{2(x+1) - 2x}{(x+1)^2}$$

(Quotient Rule for derivative)

$$(x+1)^2$$

$$= 2/(x+1)^2 > 0$$

$\Rightarrow$ $f(x)$ is increasing

$a_n = \dfrac{2n}{n+1}$ is $\left.\begin{array}{l}\text{increasing} \\ \text{\& bounded}\end{array}\right\} \Rightarrow$ convergent!

SERIES: $a_0 + a_1 + a_2 + a_3 + a_4 + \dots$

$$\sum_{n=0}^{\infty} a_n$$

Summation

If $\displaystyle\sum_{n=0}^{\infty} a_n = S$ (real number)

then the series $\displaystyle\sum_{n=0}^{\infty} a_n$ is said to converge.

example: $1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \dots$

Can we find the sum of the above series ?

$$a_0 + a_1 + a_2 + a_3 + \dots$$

$$\boxed{a_n = 1/2^n \qquad n = 0, 1, 2, \dots}$$

$a_n = 1/2^n$

Note: $\frac{a_2}{a_1} = \frac{1/4}{1/2} = 0.5$ $\quad \frac{a_3}{a_2} = \frac{1/8}{1/4} = 0.5$

$a_4/a_3 = \frac{1/16}{1/8} = 0.5$

$1 + 1/2 + 1/2^2 + \cdots$

Ratio $\frac{a_{n+1}}{a_n}$ always fixed to 0.5

$a + ar + ar^2 + ar^3 + ar^4 + \cdots$

GEOMETRIC SERIES

r = Ratio that is being r^n

$$|r| < 1 \Rightarrow \sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}.$$

same number different power

example: $\sum_{n=0}^{\infty} (3/2)^n$ find the sum of the given infinite series (if possible)

$$(3/2)^0 + (3/2)^1 + (3/2)^2 + (3/2)^3 + \cdots$$

$$\longrightarrow (3/2)^0 + (3/2)^1 + (3/2)^2 + (3/2)^3 + \cdots$$

$$1 + 1.5 + (1.5)^2 + \cdots$$

Geometric Series of form:

$$a + ar + ar^2 + ar^3 + \cdots$$

with $a = 1$ and $r = 1.5 = 3/2$

$$|r| < 1 \Rightarrow \sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

But for us $r = 1.5 > 1 \Rightarrow$ Given series

$$\sum_{n=0}^{\infty} (3/2)^n$$ Diverges.

<u>example 2:</u> $$\sum_{n=0}^{\infty} 1^n = \sum_{n=0}^{\infty} 1$$

$$1 + 1 + 1 + 1 + 1 + \cdots$$

diverges to ∞.

<u>example 3:</u> $$\sum_{n=0}^{\infty} (2/3)^n$$

$$1 + 2/3 + (2/3)^2 + (2/3)^3 + \cdots$$

$$a + ar + ar^2 + ar^3$$

$$a + ar + ar^2 + ar^3$$

with $r = 2/3$ $\quad a = 1$

$r < 1$

$$\Rightarrow \boxed{\sum_{n=0}^{\infty} ar^n = a/_{1-r}} = 1/_{1-2/3}$$

$= 3$

$$3 = \frac{1}{1/3} = \frac{1}{1-2/3}$$

$a_0 = 2$

$a_1 = 2$

$a_2 = 2$

$a_3 = 2$

$\vdots$

$\{a_n\} \to ?$

$a_n \to 2$ as $n \to \infty$

Series $a_0 + a_1 + a_2 + \cdots$

$2 + 2 + 2 + 2 + 2 + 2 \cdots \to \infty$

Geom. Series $r = 1$ $\quad a = 2$

$$2 + 2*1^1 + 2*1^2 + 2*1^3 + \cdots$$

$$a \quad ar \quad ar^2 \quad ar^3 + \cdots$$

$r = 1 \Rightarrow$ Geom. Series Diverges.