CPS 5310

Tuesday, April 21, 2020 1:39 PM

Last Lecture: $\frac{d\vec{x}}{dt} = \vec{F}(t,\vec{x})$

Goal: Long term behavior of $\vec{x}(t)$.

$\vec{F}(t,\vec{x}) = \vec{0}$?

$\vec{F}(t,\vec{x}) = A\vec{x}$ Linear vector valued function in x & y.

example: $x_1' = x_1 - 2x_2$ $\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$x_2' = 2x_1 - 3x_2$

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\hookrightarrow A$ see Notes Phase Plane critical_pts.pdf } summary cps5310-sharma/material/

λ_1, λ_2 eigen values of A determine $\vec{x}(t)$ as $t \to \infty$

for $\frac{d\vec{x}}{dt} = A\vec{x}$.

Real number values of λ_1, λ_2

$\lambda_1, \lambda_2 < 0 \Rightarrow$ asymptotic stability

other unstable

What happens if $F(t,\vec{x})$ is non linear?

example: $\begin{cases} x_1' = x_1 - 2x_2 - 1 \\ x_2' = 2x_1 - 3x_2 - 3 \end{cases}$

$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$$\frac{d\vec{x}}{dt} = \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix} \vec{x} + \begin{pmatrix} -1 \\ -3 \end{pmatrix}$$

Eqbm points:
$$x_1 - 2x_2 - 1 = 0$$
$$2x_1 - 3x_2 - 3 = 0$$

Solve for (x_1, x_2)
$$x_1 - 2x_2 = 1 \quad * -2.$$
$$2x_1 - 3x_2 = 3$$

$$-2x_1 + 4x_2 = -2$$
$$+ \quad 2x_1 - 3x_2 = 3$$
$$x_2 = 1$$

$$x_1 - 2x_2 = 1 \Rightarrow x_1 = 3$$

$(3,1)$ is an eqilibrium pt for $x_1' = x_1 - 2x_2 - 1$
$x_2' = 2x_1 - 3x_2 - 3$

RHS function $F(t, \vec{x}) = \begin{pmatrix} x_1 - 2x_2 - 1 \\ 2x_1 - 3x_2 - 3 \end{pmatrix}$
$\equiv F(\vec{x})$

Taylor expansion
Linearize $F(\vec{x})$ about the eqbm pt $x^* = (3,1)$.

$x = \vec{x}$

Jacobian

$$\boxed{F(x) = F(x^*) + JF(x^*)(x - x^*)}$$

One Dimension
$$f(x) = f(0) + f'(0)(x-0)$$ Taylor Poly Linearizatn abt 0.

$x_1' = f_1(x_1, x_2)$ $\quad x^* = (x_1^*, x_2^*)$ eqbm pt.
$x_2' = f_2(x_1, x_2)$

$$x_1' = f_1(x_1, x_2) = f_1(x_1^*, x_2^*) + \partial_{x_1} f_1(x_1^*, x_2^*)(x_1 - x_1^*)$$

$$x_1' = f_1(x_1, x_2) = f_1(x_1^*, x_2^*) + \partial_{x_1} f_1(x_1^*, x_2^*)(x_1 - x_1^*) + \partial_{x_2} f_1(x_1^*, x_2^*)(x_2 - x_2^*)$$

$$x_2' = f_2(x_1, x_2) = f_2(x_1^*, x_2^*) + \partial_{x_1} f_2(x_1^*, x_2^*)(x_1 - x_1^*) + \partial_{x_2} f_2(x_1^*, x_2^*)(x_2 - x_2^*)$$

<u>Using</u>:

$$JF(x^*) = \begin{bmatrix} \partial_{x_1} f_1(x_1^*, x_2^*) & \partial_{x_2} f_1(x_1^*, x_2^*) \\ \partial_{x_1} f_2(x_1^*, x_2^*) & \partial_{x_2} f_2(x_1^*, x_2^*) \end{bmatrix}$$

$$x' = \frac{dx}{dt} = F(x) = F(x^*) + JF(x^*)(x - x^*)$$

(in hwk04 2(c) we need to linearize.)

$$x^* = (3,1)$$

$$x' = \underset{\substack{\| \\ 0}}{F(x^*)} + JF(x^*)(x - x^*)$$

because x^* is eqbm pt i.e $\frac{dx^*}{dt} = F(x^*) = 0$

$$x' = JF(x^*)(x - x^*)$$

where $JF(x) = \begin{bmatrix} \partial_{x_1} f_1 & \partial_{x_2} f_1 \\ \partial_{x_1} f_2 & \partial_{x_2} f_2 \end{bmatrix} = \begin{bmatrix} \nabla f_1^T \\ \nabla f_2^T \end{bmatrix}$

with $f_1(x_1, y_1) = x_1 - 2x_2 - 1 \rightarrow \partial_{x_1} f_1 = 1 \quad \partial_{x_2} f_1 = -2$

with $f_1(x_1, y_1) = x_1 - 2x_2 - 1 \to \partial_{x_1} f_1 = 1$ $\quad \partial_{x_2} f_1 = -2$

$f_2(x_1, y_1) = 2x_1 - 3x_2 - 3 \to \partial_{x_1} f_2 = 2$ $\quad \partial_{x_2} f_2 = -3$

$$JF(x) = \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}$$ const Jacobian

$JF(x^*)$

$$\left.\begin{aligned} x_1' &= x_1 - 2x_2 - 1 \\ x_2' &= 2x_1 - 3x_2 - 3 \end{aligned}\right\} \to \text{non linear RHS}$$

$x^* = (3, 1)$

$$X' = F(x^*) + JF(x^*)(x - x^*)$$

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = x' = \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix} \begin{pmatrix} x_1 - 3 \\ x_2 - 1 \end{pmatrix}$$

make a transformation $\quad \hat{x}_1 = x_1 - 3$

$\hat{x}_2 = x_2 - 1$

$x_1' = \hat{x}_1'$

$x_2' = \hat{x}_2'$

$$\underbrace{\begin{bmatrix} \hat{x}_1' \\ \hat{x}_2' \end{bmatrix}}_{\hat{x}'} = \begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \end{bmatrix}}_{\hat{x}}$$

$$\hat{x}' = A\hat{x}$$

determine behavior of (x_1, x_2) by eigenvalues of A.

$\begin{bmatrix} 1 & -2 \end{bmatrix}$ e.values: $\begin{vmatrix} 1-\lambda & -2 \end{vmatrix} = 0$

of A.

$$A = \begin{bmatrix} 1 & -2 \\ 2 & -3 \end{bmatrix}$$ e.values: $|A - \lambda I| = 0$ $\quad \begin{vmatrix} 1-\lambda & -2 \\ 2 & -3-\lambda \end{vmatrix} = 0$

$$(\lambda-1)(\lambda+3) - (-4) = 0$$

$$\lambda^2 + 2\lambda - 3 + 4 = 0$$

$$\lambda^2 + 2\lambda + 1 = 0$$

$$\lambda = -1, -1$$

evalues are negative hence $(3,1)$ is an asymptotically stable equilibrium pt.

Pop^n models → interaction of species (Predator-Prey model)

Prey $X(t)$ Predator pop^n $Y(t)$ $a, b, c, d > 0$ consts

$$X' = aX - cXY \rightarrow XY$$

$$Y' = -bY + dXY$$

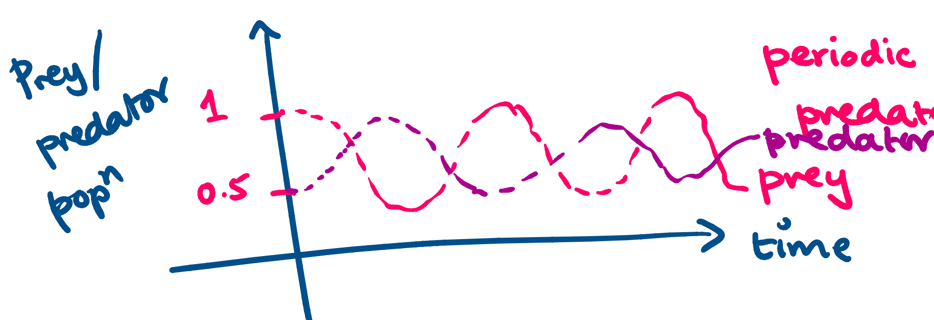
XY "interaction terms" "coupling"

Note: $\begin{cases} c = 0 \Rightarrow & X(t) = x_0 e^{at} \\ d = 0 \Rightarrow & y(t) = y_0 e^{-bt} \end{cases}$

relation & impact of prey pop^n on predator pop^n & v.v.

example: $\begin{cases} x' = \overset{a}{4}x - \overset{c}{2}xy & x(0) = 1 \text{ prey} \\ y' = -\overset{b}{3}y + \overset{d}{3}xy & y(0) = 0.5 \end{cases}$

$h = \Delta t = 0.001$ accurate soln



periodic evolution of predator & prey populatns

(c...it... for

Phase Plane/Portraits for

$$x' = 4x - 2xy$$
$$y' = -3y + 3xy$$

eqbm popⁿs: $x' = 0$ $y' = 0$

$x' = 0 \Rightarrow 4x - 2xy = 0 \Rightarrow x = 0$ or $y = 2$

$x = 0$ or $x \neq 0$ if $x \neq 0$

then, divide by $x \neq 0$

$$\frac{x}{x}(4 - 2y) = 0$$

$$4 - 2y = 0$$

$$\Rightarrow y = 2$$

$y' = 0 \Rightarrow -3y + 3xy = 0 \Rightarrow y = 0$ or $y \neq 0$ and

$$-3 + 3x = 0$$
$$\boxed{x = 1}$$

Eqbm points: $\boxed{(0,0) \quad (1,2)}$, $(0,2)$, $(1,0)$

$(0,0)$

$$X' = F(X) \qquad X = \begin{pmatrix} x \\ y \end{pmatrix}$$

$X^* = (0,0)$

$$X' = F(X^*) + JF(X^*)(X - X^*)$$

$$= 0 + JF(X^*)\, X$$

$JF(X) = \begin{bmatrix} \partial_x f_1 & \partial_y f_1 \end{bmatrix}$ $\quad f_1 = 4x - 2xy$

$$JF(x) = \begin{bmatrix} \partial_x f_1 & \partial_y f_1 \\ \partial_x f_2 & \partial_y f_2 \end{bmatrix} \qquad \begin{aligned} f_1 &= 4x - 2xy \\ f_2 &= -3y + 3xy \end{aligned}$$

$$JF\left((x,y)\right) = \begin{bmatrix} 4-2y & -2x \\ 3y & -3+3x \end{bmatrix}$$

$x^* = (1,2)$ (do $(0,0)$ later)

$x^* = (1,2)$

$$JF(x^*) = \begin{bmatrix} 0 & -2 \\ 6 & 0 \end{bmatrix}$$

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{bmatrix} 0 & -2 \\ 6 & 0 \end{bmatrix} \begin{bmatrix} x-1 \\ y-2 \end{bmatrix}$$

$\hat{x} = x - 1$; $\hat{y} = y - 2$

$$\begin{pmatrix} \hat{x}' \\ \hat{y}' \end{pmatrix} = \begin{bmatrix} 0 & -2 \\ 6 & 0 \end{bmatrix} \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix}$$

$$\hat{X}' = A\hat{X} \qquad \hat{X} = \begin{bmatrix} \hat{x} \\ \hat{y} \end{bmatrix}$$

eigen values of A: $|A - \lambda I| = 0$

$$\begin{vmatrix} 0-\lambda & -2 \\ 6 & 0-\lambda \end{vmatrix} = 0$$

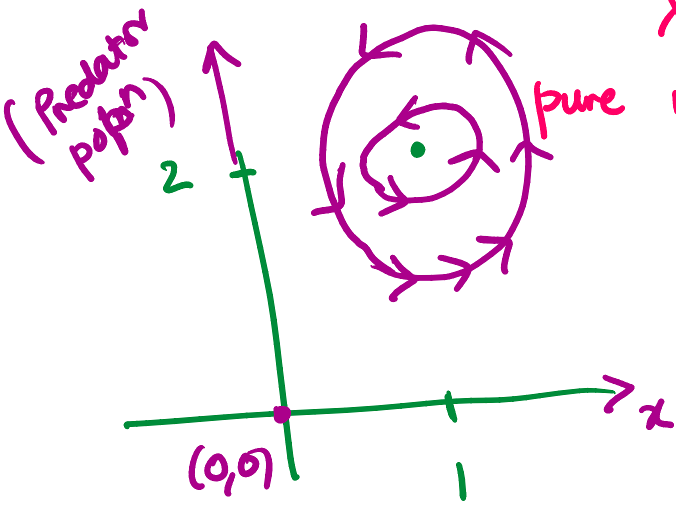
find: $\lambda^2 - (-12) = 0$

$\lambda^2 + 12 = 0$

$$\lambda^2+12 = 0$$

$$\lambda = \pm i\sqrt{12}$$

pure imaginary solns



Stable but not asymp. Stable

"circling the source sink point)

exam → 1:00pm → 24th April 11:59pm

email exam01 to nsshama@utep.edu

↳ programming MATLAB.

1:30 - 2:50 pm "office hours"

OPEN BOOK exam