CPS-5310-online

Tuesday, April 7, 2020 11:24 AM

Phase Lines → Predicting the long term behavior.

$y(t)$ $\qquad \frac{dy}{dt} = f(y,t)$; $y(0) = y_0$ (IVP)

Ques: As $t \to \infty$, $y(t) \to$??

$y(t) \to \pm\infty$? $|y| \to +\infty$

or $y(t) \to$ one of its equilibrium states?

Answer to this is determined by:

Data of the problem
(1) y_0
(2) $f(t,y)$

Long Term Behavior of Logistic Growth Model?

Example: $\frac{dy}{dt} = \underbrace{0.5\,y\left(1 - \frac{y}{10}\right)}_{f(t,y)}$ — $0.5 = r$; 10 → carrying capacity

Equilibrium populations:

$\frac{dy}{dt} = 0 \Rightarrow y = 0,\ y = 10$

$\frac{dy}{dt} = f(t,y) > 0 \qquad 0 < y < 10$

$f(t,y) = 0 \qquad y = 0,\ y = 10$

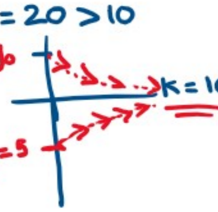
$f(t,y) < 0 \qquad y > 10,\ y < 0$ (Non Physical soln)

Long Term Behavior → ? Phase Line $\qquad \frac{dy}{dt} = 0.5\,y(1 - y/10)$

choice of y_0:

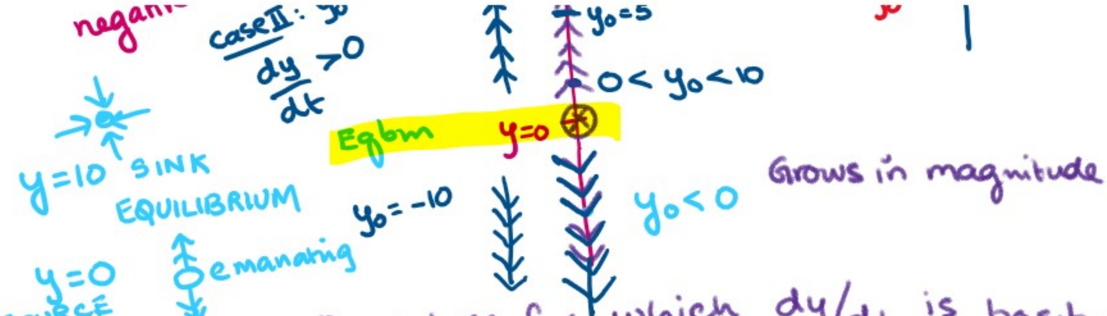
$\frac{dy}{dt} = 0.5y\underbrace{(1 - y/10)}_{\text{negative at } y>10}$

case I $\quad y_0 = 20 > 10$



case II: $y_0 = 5$

$\frac{dy}{dt} > 0$



Knowing pop^n values for which dy/dt is positive or negative is crucial for predicting long term behavior.

$$\frac{dy}{dt} = 0.5\left(1 - y/10\right)y$$

Conclusion: $y_0 < 0 \Rightarrow y(t) \to -\infty$ (Non-phy. Solution)

$0 < y_0 < 10 \Rightarrow y(t) \to 10$ increases to 10

$y_0 \geq 10 \Rightarrow y(t) \to 10$ decreases to 10

Revisit Phase Lines when we consider sys. of ODES.

Parameter Estimation:

① Visual plot the data

② Best fit curve for the data

③ Numerical Derivative Line fit

Linear* fit

$y = mx + b$

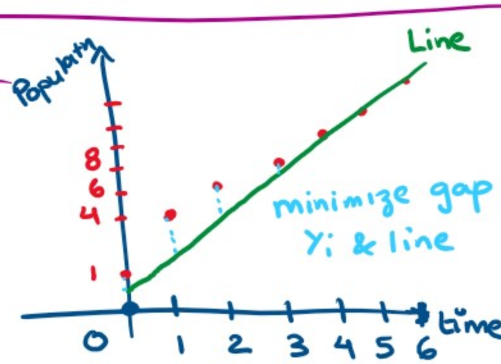
Nonlinear fit

- Quadratic* fit
- periodic fit
- Analytic Solution fit

*MATLAB TOOL: polyfit(x, y, poly degree)

returns $p(1)x^n + p(2)x^{n-1} + \ldots + p(n+1)$, $n =$ poly degree

x_i	Time in minutes	Population y_i
1	0	1
2	10	4
3	20	6
4	30	8
5	40	10
		12



5	40	10
6	50	12
7	60	14

0 1 2 3 4 5 6 time

Goal : make predictions about popⁿ growth based on my observations.

Natural Approach → Construct a curve that passes through (as close as possible) the given data.

"BEST FIT LINE" $P(x_i) = y_i$ Relaxing this condition!

What kind of a curve should we construct?

* Line

$(x_1, y_1), \ldots (x_7, y_7)$ → given discrete data.

Goal : $y = mx + b$ find m & b so that instead of demanding $y_i = mx_i + b$

we require $\sum_{i=1}^{7} |y_i - (mx_i + b)|^2$ is minimized! Least squares method

$$E(m,b) = \sum_{i=1}^{7} |y_i - (mx_i+b)|^2$$

$$\min_{m,b} E(m,b)$$

From Calculus, $\boxed{\frac{\partial E}{\partial b} = 0}$ $\boxed{\frac{\partial E}{\partial m} = 0}$

2 equations in 2 unknowns m, b.

solve for m & b.

$$E(m,b) = \sum_{i=1}^{7} (y_i - mx_i - b)^2$$

solve for m & b,

$$\frac{\partial E}{\partial m} = 0 \Rightarrow \sum_{i=1}^{7} 2(y_i - mx_i - b)(-x_i) = 0$$

$$\frac{\partial E}{\partial m} = 0 \Rightarrow \sum_{i=1}^{7} 2(y_i - mx_i - b)(-x_i) = 0$$

$$(-2)\sum_{i=1}^{7}(x_i y_i - m x_i^2 - b x_i) = 0$$

$$\sum_{i=1}^{7} x_i y_i - m\sum_{i=1}^{7} x_i^2 - b\sum_{i=1}^{7} x_i = 0$$

(x_i, y_i) given!

$$① \leftarrow \boxed{\left(\sum_{i=1}^{7} x_i^2\right) m + \left(\sum_{i=1}^{7} x_i\right) b = \sum_{i=1}^{7} x_i y_i}$$

$$E(m,b) = \sum_{i=1}^{7} |y_i - mx_i - b|^2$$

$$\frac{\partial E}{\partial b} = 0 \Rightarrow \sum_{i=1}^{7} 2(y_i - mx_i - b) * (-1) = 0$$

$$(-2)\left(\sum_{i=1}^{7} y_i - m\left(\sum_{i=1}^{7} x_i\right) - b\sum_{i=1}^{7} 1\right) = 0$$

$(x_i y_i)$ are known,

$$② \leftarrow \boxed{\left(\sum_{i=1}^{7} x_i\right) m + 7b = \sum_{i=1}^{7} y_i}$$

multiply ② by $\left(\sum_{i=1}^{7} x_i\right) / 7$ and subtract from ①.

and solve for m.

$$m = \frac{\sum_{i=1}^{7} x_i y_i - \left(\sum_{i=1}^{7} x_i\right)\left(\sum_{i=1}^{7} y_i\right) / 7}{\sum_{i=1}^{7} x_i^2 - \left(\sum_{i=1}^{7} x_i\right)^2 / 7}$$

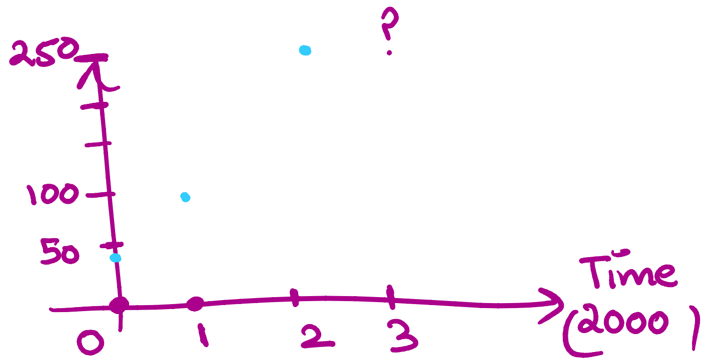
substitute m to obtain b,

$$b = 1/7 \sum_{i=1}^{7} y_i - \left(\sum_{i=1}^{7} x_i\right) m$$

$b = 1/7 \sum_{i=1} y_i \quad \left(\sum_{i=1} \right) \ldots$

Polyfit

Year	Net Income
2000	48.3 million
2001	90.4 million
2002	249.9 million



Based on the above net revenue, predict the net income for the year 2003.

Task: Construct a line that fits the given data

x_i	y_i	$\Sigma x_i y_i$	Σx_i^2
0	48.3	0	0
1	90.4	90.4	1
2	249.9	499.8	4
$\Sigma x_i = 3$	388.6	590.2	5

Σy_i

formula:

$$m = \frac{\sum_{i=1}^{n} x_i y_i - 1/n \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n} y_i\right)}{\sum_{i=1}^{n} x_i^2 - 1/n \left(\sum_{i=1}^{n} x_i\right)^2}$$

$$m = \frac{590.2 - 1/3 \quad 3 * 388.6}{5 - 3} = \frac{201.6}{2} = 100.8$$

formula

$$b = \frac{1}{n}\left(\sum_{i=1}^{n} y_i\right) - \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right) m$$

$$1 \quad 388.6 - 1 * 3 * 100.8$$

$$= \frac{1}{3}\ 388.6 - \frac{1}{3} * 3 * 100.8$$

$$b = 28.73$$

Try

$$p = \text{polyfit}(x, y, 1)$$

$$p = [\underbrace{100.8}_{m},\ \underbrace{28.73}_{b}]$$