CPS-5310-online

Tuesday, March 31, 2020 1:27 PM

Last class:

$$y' = f(t,y)$$
$$y(0) = y_0$$

2 methods to analytically solve the ODE. formula $y(t)$

if $f(t,y) = g(t)\,h(y)$

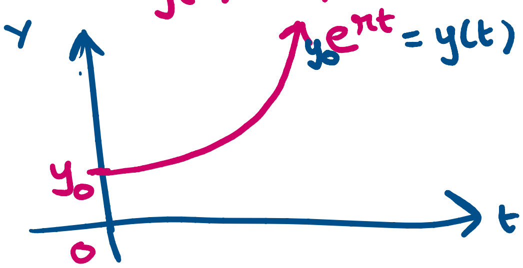
then, $\displaystyle\int \frac{dy}{h(y)} = \int g(t)\,dt$

Tank mixing problem.

Population models:

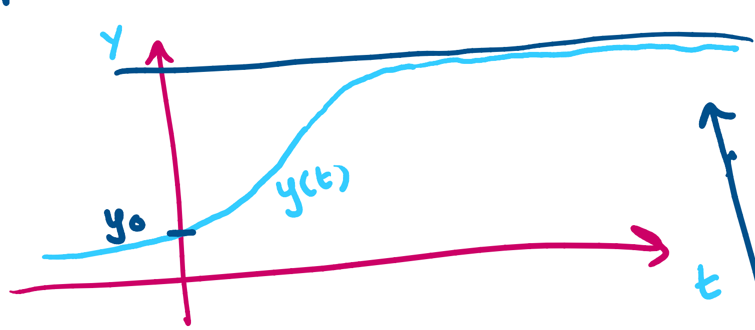
$$\boxed{y' = ry}$$
$$y(0) = y_0$$

$y(t)$: Pop^n at time t



$$\left.\begin{aligned} y' &= ry \\ y(0) &= y_0 \end{aligned}\right\} \rightarrow \text{solution } y(t) = y_0 e^{rt}$$

exponential pop^n growth



carrying capacity $y = K$

$y(t) \to K$ as $t \to \infty$

$r \to$ rate of Growth of the populatn

Logistic Growth Model

$$y' = ry\,(1 - y/K) \qquad y \equiv y(t)$$

Logistic Growth Model

$$y' = ry\left(1 - y/K\right) \qquad y \equiv y(t), \quad y' \equiv y'(t)$$

Notice that there are 2 population "states" where we have no change $y' = 0$.

$$ry\left(1 - y/K\right) = 0$$

$y = 0$ or $y = K$ Populations where equilibrium is achieved

Goal lecture:

1. find the closed form solution to $y' = ry\left(1 - y/K\right)$
2. Numerical methods to solve ODE

$$\boxed{\begin{aligned} y' &= ry\left(1 - y/K\right) \\ y(0) &= y_0 \end{aligned}}$$

use partial fraction decomposition

step 1: $y' = \dfrac{dy}{dt} = \overset{\text{Rate}}{r}y\left(1 - y/K\right)$

$$\int \frac{dy}{y\left(1 - y/K\right)} = \int r\,dt$$

Note: holds if $1 - y/K \neq 0$ and $y \neq 0$.

$$\int \frac{dy}{y\left(1 - y/K\right)} = \int r\,dt$$

$$\int \frac{}{y(1-y/K)} \qquad\qquad \int$$

$$\frac{1}{y(1-y/K)} = \frac{A}{y} + \frac{B}{1-y/K} \Rightarrow \text{multiply both sides by } y(1-y/K)$$

$$1 = A(1-y/K) + By$$

$$y=K \Rightarrow B = 1/K$$

$$y=0 \Rightarrow A = 1$$

$$\int \frac{dy}{y(1-y/K)} = \int \left(\frac{A}{y} + \frac{B}{1-y/K} \right) dy$$

$$= \int \left(\frac{1}{y} + \frac{1/K}{1-y/K} \right) dy$$

$$= \int \frac{1}{y} dy + \frac{1}{K} \int \frac{dy}{\underbrace{1-y/K}}$$

$\hookrightarrow$ let $u = 1-y/K$

$du = -dy/K$

$$= \ln|y| - \int \frac{du}{u}$$

$$= \ln|y| - \ln|1-y/K|$$

$\ln A - \ln B = \ln\left(\frac{A}{B}\right)$

$$= \ln \left| \frac{y}{1-y/K} \right|$$

$$\int \frac{dy}{y(1-y/K)} = r \int dt$$

$$\downarrow$$

$$\ln \left| \frac{y}{1-y/K} \right| = rt + c$$

$$\ln \left| \frac{y}{1-y/K} \right| =$$

$$e^{\ln \left| y/(1-y/K) \right|} = e^{rt+c}$$

$$\left| \frac{y}{1-y/K} \right| = e^{rt+c}$$

Assume $y>0$ $\quad 1-y/K > 0$

$$\frac{y}{1-y/K} = e^{rt+c}$$

$C = ?$ $\quad y(0) = y_0$

$$\frac{y_0}{1-y_0/K} = e^{r*0+c}$$

$$\frac{y_0}{1-y_0/K} = e^c$$

$$y/_{1-y/K} = e^{rt} \cdot e^c = e^{rt} * \frac{y_0}{1-y_0/K}$$

$$y(t) = \boxed{?}$$

Currently, $\dfrac{y}{1-y/K} = e^{rt} * \dfrac{y_0}{1-y_0/K}$

mult. by $(1-y/K)$

$$y = e^{rt} \frac{y_0}{1-y_0/K} \left(1-y/K\right)$$

$$y = e^{rt}\frac{y_0}{1-y_0/K}\left(1-y/K\right)$$

$$\boxed{y} = e^{rt}\frac{y_0}{1-y_0/K} - \frac{y e^{rt} y_0}{K(1-y_0/K)}$$

add $\dfrac{ye^{rt}y_0}{K(1-y_0/K)}$ to both sides.

$$\left(1 + \frac{e^{rt}y_0}{\underbrace{K(1-y_0/K)}_{(K-y_0)}}\right)y = \frac{e^{rt}y_0}{1-y_0/K}$$

$$\left(1+\frac{e^{rt}y_0}{K-y_0}\right)y = \frac{e^{rt}y_0}{1-y_0/K} = \frac{Ke^{rt}y_0}{K-y_0}$$

$$y(t) = \frac{Ke^{rt}y_0/K-y_0}{1+e^{rt}y_0/K-y_0} = \frac{Ky_0}{y_0+(K-y_0)e^{-rt}}$$

Parameter estimation

$$y(t) = \frac{Ky_0}{K+(K-y_0)e^{-rt}}$$

t	y(t)
1900	1
1910	10
1920	20
⋮	⋮

Numerical Solutions to ODE: Gompertz equation

$$y' = -r\ln\left(\frac{y}{K}\right)y$$

$$y(0) = y_0$$

$$y(0) = y_0$$

$r \rightarrow$ rate of growth $\quad k \rightarrow$ carrying capacity of pop^n

Numerical method to solve:

$$y' = -r\ln\left(y/k\right) y$$

Euler Method: Replace $y'(t) \rightarrow D_h y(t)$, h = time step

$$= \frac{y(t+h) - y(t)}{h}$$

$$\frac{y(t+h) - y(t)}{h} = -r\ln\left(\frac{y(t)}{k}\right) y(t)$$

Solve for y at time $t+h$ given $y(t)$.

$$y(t+h) = y(t) + h * \left(-r\ln\left(\frac{y(t)}{k}\right) y(t)\right)$$