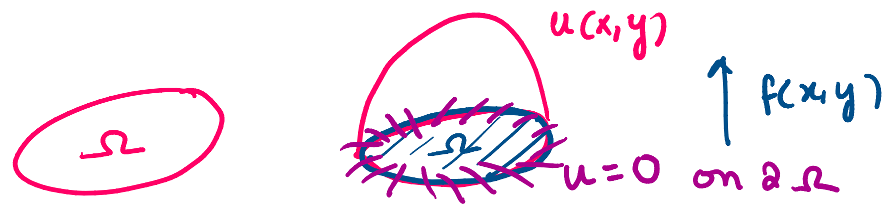
Thursday, May 7, 2020 1:34 PM

Last Lecture: $E(u)$ = energy associated with system



$$E(u^*) = \min_{w \in V} E(w)$$

iff $$\lim_{\lambda \to 0} \frac{E(u^* + \lambda v) - E(u^*)}{\lambda} = 0 \quad \forall v \in V$$

Optimality Condn

Gateaux Deri

$\boxed{f'(x)=0} \to \langle E'(u^*); v\rangle = 0 \quad \forall v \in V$

switch notation from $E(v)$ to $J(u)$

for our energy:

$$\langle J'(u); v\rangle = \boxed{\mu \int_\Omega \nabla u \cdot \nabla v \, dx - \int_\Omega f v \, dx}$$

$$\lim_{\lambda \to 0} \frac{J(u+\lambda v) - J(u)}{\lambda}$$

Divergence Thm

using $$\int_\Omega \nabla u \cdot \nabla v \, dx = -\int_\Omega \Delta u \, v \, dx + \int_{\partial\Omega} \frac{\partial u}{\partial n} v \, ds$$

where $\Delta u = \nabla \cdot (\nabla u)$ $\quad \partial u/\partial n = \hat{n} \cdot \nabla u$

Optimality Condn: $\langle J'(u); v\rangle = 0$ becomes

$$\mu \int_\Omega \nabla u \cdot \nabla v \, dx - \int_\Omega f v \, dx = 0 \quad \forall v \in V$$

$$\mu\left(-\int_\Omega \Delta u \, \underline{\underline{v}} \, dx\right) - \int_\Omega f \underline{\underline{v}} \, dx = 0 \quad \forall \underline{\underline{v \in V}}$$

$$\int_\Omega (-\mu \Delta u - f) \, v \, dx = 0 \quad \forall v \in V$$

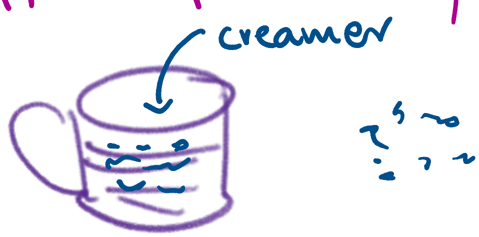
$$v(x) = \begin{cases} 1 & \text{inside } \Omega \\ 0 & \text{on } \partial\Omega \end{cases}$$

$$\int_\Omega$$

$$\Rightarrow \left\{ \begin{array}{l} -\mu\Delta u - f \quad = 0 \\ -\mu \Delta u = f \;;\quad u = 0 \text{ on } \partial\Omega. \end{array} \right\}$$

$$\boxed{\begin{array}{l} -\mu \nabla\cdot\nabla u = f \text{ in } \Omega \\ u = 0 \text{ on } \partial\Omega \end{array}}$$ → POISSON EQUATION

Diffusive process $\quad \mu \to$ diffusion coeff



#2 PDE Activity

Given $\quad J(v) = \dfrac{1}{2}\displaystyle\int_\Omega |\nabla v|^2 dx - \int_\Omega g v \, dx$

0 $\quad \Omega \quad$ 1

$\Omega \subseteq \mathbb{R} \Rightarrow \quad v : \Omega \to \mathbb{R} \qquad \Omega = (0,1)$

$$\nabla v = \frac{\partial v}{\partial x} = \frac{dv}{dx} = v'$$

$$J(v) = \frac{1}{2}\int_0^1 (v')^2 dx - \int_0^1 g v \, dx$$

$$\min_{v \in V} J(v) = J(u), \quad \text{show } u = -\frac{g}{2}(x^2 - x)$$

optimality condn $\quad \langle J'(u); v\rangle = 0 \quad \forall v \in V$

$$\lim_{\lambda \to 0} \frac{J(u+\lambda v) - J(v)}{\lambda} = 0 \qquad \forall v \in V$$

$$\lim_{\lambda \to 0} \frac{J(u+\lambda v) - J(v)}{\lambda} =$$

$$J(u+\lambda v) = \frac{1}{2}\int_0^1 ((u+\lambda v)')^2 dx - \int_0^1 g(u+\lambda v) dx$$

$$\hat{J}(u+\lambda v) = \frac{1}{2}\int_0^1 ((u+\lambda v)')^2 dx - \int_0^1 g(u+\lambda v)dx$$

$$= \frac{1}{2}\int_0^1 (u'+\lambda v')^2 dx - \int_0^1 gu + \lambda gv \, dx$$

$$\frac{J(u+\lambda v) - J(u)}{\lambda} = \frac{1}{2\lambda}\int_0^1 ((u')^2 + 2u'\lambda v' + (v')^2 - (u')^2 dx$$

$$- \frac{1}{\lambda}\int_0^1 g(u+\lambda v) - gu \; dx$$

$$\lim_{\lambda\to 0} \frac{J(u+\lambda v) - J(u)}{\lambda} = \int_0^1 u'v' dx - \int_0^1 gv\, dx$$ Gateaux Derivative

$\langle J'(u);v \rangle = 0 \quad \forall v \in V$ optimality condn

Show that given $u = -g/_2(x^2 - x)$ satisfies

$$\boxed{\int_0^1 \underline{\underline{u'}} v' dx - \int_0^1 gv\, dx = 0 \quad \forall v \in V}$$

apply int. by parts to 1st integral & use $\underline{v(0) = v(1) = 0}$

$$-\int_0^1 u'' v \, dx + u'(1)\underbrace{v(1)}_{=0} - u'(0)\underbrace{v(0)}_{=0} - \int_0^1 gv\, dx = 0$$

$$-\int_0^1 u'' \underline{\underline{v}} \, dx - \int_0^1 g\underline{\underline{v}} dx = 0 \quad \forall v \in V$$

identical

$$-\int_0 u'' \underline{\underline{v}}\, dx = \int_0 g \underline{\underline{v}}\, dx$$

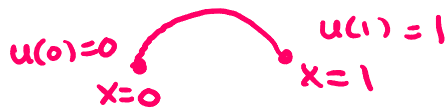
or $\boxed{\text{PDE}}$

$$\left.\begin{aligned} -u'' &= g \text{ in } \Omega \\ u(0) = u(1) &= 0 \end{aligned}\right\}$$

compare to 2D identical

$$-\Delta u = g \text{ in } \Omega$$
$$u = 0 \text{ on } \partial\Omega$$

Verify $u(x) = -g/_2(x^2 - x)$ satisfies PDE and boundary condn $u(0) = 0$, $u(1) = 0$



$$u''(x) = -g\left(\tfrac{2}{2}\right) = -g$$

$u(x)$ is indeed a minimizer of energy

$$J(v) = \frac{1}{2}\int_0^1 (v')^2\, dx - \int_0^1 g v\, dx.$$

$$\langle J'(u); v\rangle = 0 \longrightarrow \text{PDE} \quad -u'' = g, \quad u(0) = u(1) = 0$$

V = space of permissible displacements

V = space of polynomials of degree atmost 2 defined on $[0,1]$ & which are zero at 0 & 1

III. $$J(v) = \frac{1}{2}\int_\Omega \left(\frac{d^2v}{dx^2}\right)^2 dx - \int_\Omega f v\, dx$$

Given $\Omega = (0, L)$

$f(x) = f_0$

Shorthand $v'' = \dfrac{d^2v}{dx^2}$

$$J(v) = \frac{1}{}\int^L (v'')^2\, dx - \int^L f_0 v\, dx$$

$$J(v) = \frac{1}{2} \int_0 (v'')^2 \, dx - \int_0 f_0 v \, dx$$

Task: $\min_{v \in V} J(v) = J(u)$

1. $V = ?$ = space of acceptable displacements V

$$= \left\{ v : \Omega \to \mathbb{R} \text{ such that } v(x) = 0 \text{ at } x = 0, L; \; J(v) = \frac{1}{2} \int_0^L (v'')^2 dx - \int_0^L f v \, dx \text{ is well defined} \right\}$$

2. show $\boxed{J(u) = \min_{v \in V} J(v)}$ has ! soln

$J(v)$ strictly convex $\Rightarrow$ u is unique.

3. $\langle J'(u); v \rangle = 0$ (Optimality Condn) for all $v \in V$.

$$J(u) = \frac{1}{2} \int_0^L (u'')^2 \, dx - \int_0^L f_0 u \, dx$$

$$\langle J'(u); v \rangle = \int_0^L u'' v'' \, dx - \int_0^L f_0 v \, dx$$

apply integration by parts

$v'(0) = v'(1) = 0$

apply integration by parts twice & use $v'(0) = v'(1) = 0$
$v(0) = v(1) = 0$

Opt. condn:

$$\int_0^L u'' v'' \, dx - \int_0^L f_0 v \, dx = 0 \quad \forall v \in V$$

$$\int_0^L u'''' \underline{\underline{v}} \, dx - \int_0^L f_0 \underline{\underline{v}} \, dx = 0 \quad v \in V$$

$$\int_0^L (u'''' - f_0) \, v \, dx = 0 \quad \forall v \in V$$

In part. take $v = \begin{cases} 1 & \text{in } (0,L) \\ 0 & \text{at } x = 0, L. \end{cases}$

$$\int_0^L (u'''' - f_0) \, dx = 0 \Rightarrow u'''' - f_0 = 0$$

$$u'''' = f_0 \text{ in } \Omega$$

$$\left\{ \begin{array}{l} u'(0) = u'(L) = 0 \\ u(0) = u(L) = 0 \end{array} \right\}$$

check if given $u(x) = \frac{f_0}{24}(x^4 - 2Lx^3 + L^3 x)$

satisfies Opt. condn $\rightarrow \left\{ \begin{array}{l} u'''' - f_0 = 0 \\ u'(0) = u'(L) = 0 \\ u(0) = u(L) = 0 \end{array} \right\}$

$u(0) = u(L) = 0$? check it is true ✓

$u(0) = u(L) = 0$? check it is true ✓

$u'(0) = u'(L) = 0$ ✓ check it is true ✓

$$u''''(x) = \frac{d^4}{dx^4}\left(\frac{f_0}{24}(x^4 - 2Lx^3 + L^3x)\right)$$

$$= f_0$$

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \int_\Omega f v\,dx$$

opt. condn $\quad \int_\Omega \nabla u \cdot \nabla v\,dx - \int_\Omega f v\,dx = 0$

opt. condn $\quad \boxed{\int_\Omega \nabla u \cdot \nabla v\,dx = \int_\Omega f v\,dx \quad \forall\, v \in V}$

$v \in V_h = \dim n \rightarrow \{v_1, v_2, \ldots v_n\}$

$u \in V_h$

$$\left.\begin{aligned} \int_\Omega \nabla u \cdot \nabla v_i\,dx &= \int_\Omega f v_i\,dx \\ \sum \alpha_j \int_\Omega \nabla v_j\, \nabla v_i\,dx &= \int f v_i\,dx \end{aligned}\right\} \quad i = 1, 2, \ldots n$$

$u(x) = \sum \alpha_j v_j$

$$A\vec{x} = \vec{b}$$

$$A = \left(\int_\Omega \nabla v_j \nabla v_i \, dx\right)_{ij}$$

$$\vec{x} = \begin{bmatrix}\alpha_1\\ \vdots \\ \alpha_n\end{bmatrix} \qquad \vec{b} = \begin{bmatrix}\int_\Omega f v_1 \\ \vdots \\ \int_\Omega f v_n\end{bmatrix}$$

Finite element method