CPS-5310

Tuesday, May 5, 2020 1:24 PM

Goal: $J(u) = \frac{\mu}{2}\int_\Omega |\nabla u|^2\,dx - \int_\Omega f u\,dx$

Here: $\nabla u = \begin{pmatrix} \partial_1 u \\ \partial_2 u \end{pmatrix} \equiv \begin{pmatrix} \partial_{x_1} u \\ \partial_{x_2} u \end{pmatrix}$

$\Omega \subseteq \mathbb{R}^2$ Ω could be a circle or unit square.

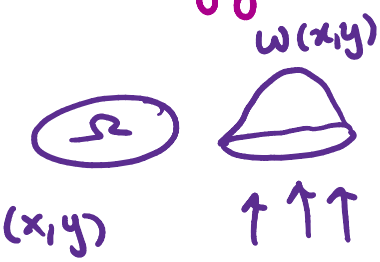
$f(x,y) \to$ external force acting on Ω.

$\mu \to$ constant response of membrane to force.

u is displacement that minimizes energy.

minimize $J(v) = J(u)$

$v \in V$



$V \to$ space of all displacements $w(x,y)$

$w(x,y) = 0$ on $\partial\Omega$ clamped on boundary $\partial\Omega$.

$$J(w) = \mu/2 \int_\Omega |\nabla w|^2\,dx - \int_\Omega f w\,dx$$

$$V \to \left\{ w(x,y) : \int_\Omega |\nabla w|^2\,dx < +\infty \quad \int_\Omega f w\,dx < +\infty \text{ and } w = 0 \text{ on } \partial\Omega \right\}$$

Goal: minimizing $J(v)$ over $v \in V$

Optimality Condition

$*J'(v) = 0^*$ verify that $J''(v) > 0$

$\Rightarrow$ v is a minizer

... mean by $J'(v) = ?$

→ what do we mean by $J'(v) = ?$

Ans: $\langle J'(v); w\rangle$ = Gateaux Derivative wrt w where $w \in V$.

$$\langle J'(v); w\rangle = \lim_{\lambda \to 0} \frac{J(v+\lambda w) - J(v)}{\lambda}$$

Calculus to $J'(v) = 0 \to \langle J'(v); w\rangle = 0$ for all $w \in V$

Generalize $J''(v) > 0 \longrightarrow$ Verifying that $J(v)$ is strictly convex

Optimality Condn:

$$\lim_{\lambda \to 0} \frac{J(v+\lambda w) - J(v)}{\lambda} = 0 \quad \text{for all } w \in V$$

By construction, $J(v)$ is strictly convex functional.

so any point v^* such that:

$$\lim_{\lambda \to 0} \frac{J(v^* + \lambda w) - J(v^*)}{\lambda} = 0 \quad \text{for all } w \in V$$

automatically becomes a minimizer for $J(v)$.

It turns out that optimality Condn

$$\lim_{\lambda \to 0} \frac{J(v^* + \lambda w) - J(v^*)}{\lambda} = 0$$

gives us the PDE describing the system.

for $J(u) = \mu/2 \int_\Omega |\nabla u|^2 dx - \int_\Omega f u \, dx$,

$\Omega \qquad \Omega$

the optimality condition is: u^* minimizes $J(u)$ if & only if it satisfies

$$\boxed{\mu \int_\Omega \nabla u^* \cdot \nabla w \, dx = \int_\Omega f w \, dx \quad \text{for all } w \in V.}$$

$\Downarrow$

$$\boxed{\begin{aligned} -\nabla \cdot (\mu \nabla u) &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned}}$$ PDE

Optimality Condition for $J(u)$: $\lim_{\lambda \to 0} \frac{J(u+\lambda v) - J(v)}{\lambda} = 0$ for all $v \in V$.

$$\frac{J(u+\lambda v) - J(u)}{\lambda} = \,?$$

claim: $\lim_{\lambda \to 0} \frac{J(u+\lambda v) - J(u)}{\lambda} = 0$ for all $v \in V$

turns out to be:

$$\mu \int_\Omega \nabla u \nabla v \, dx - \int_\Omega f v \, dx = 0 \quad \text{for all } v \in V.$$

Proof: $J(u) = \mu/_2 \int_\Omega |\nabla u|^2 dx - \int_\Omega f u \, dx$

$J(u+\lambda v)$

$$= \mu/_2 \int_\Omega \underbrace{|\nabla u + \lambda \nabla v|^2}\, dx - \int_\Omega f(u+\lambda v)\, dx$$

$$= \mu/2 \int_\Omega \underbrace{|\nabla u + \lambda \nabla v|}\, dx - \int_\Omega \ldots$$

$$= \mu/2 \int_\Omega (\nabla u + \lambda \nabla v)^T (\nabla u + \lambda \nabla v)\, dx - \int_\Omega (fu + \lambda f v)\, dx$$

T → transpose of a vector

$$= \mu/2 \int_\Omega (\nabla u^T + \lambda \nabla v^T)\, \nabla u + (\nabla u^T + \lambda \nabla v^T)\, \lambda \nabla v \quad dx$$

$$- \int_\Omega (fu + \lambda f v)\, dx$$

$J(u+\lambda v)$

$$= \mu/2 \int \nabla u^T \nabla u + 2\, \nabla v^T \nabla u + \lambda^2 \nabla v^T \nabla v \quad dx$$

$$- \int_\Omega (fu + \lambda f v)\, dx$$

↳ used: $\nabla v^T \nabla u = (\nabla v^T \nabla u)^T = \nabla u^T \nabla v$

$$\boxed{J(u) = \mu/2 \int_\Omega |\nabla u|^2 dx - \int_\Omega fu\, dx}$$

$$\frac{J(u+\lambda v) - J(u)}{\lambda} = ?$$

Simplify

$$J(u+\lambda v) = \mu/2 \int_\Omega |\nabla u|^2 + 2\lambda \nabla v^T \nabla u + \lambda^2 |\nabla v|^2 dx$$

$$- \int_\Omega fu + \lambda f v\, dx$$

$$J(u+\lambda v) - J(u) = \mu/2 \int_\Omega \cancel{|\nabla u|^2} + 2\lambda \nabla v^T \nabla u + \lambda^2 |\nabla v|^2 dx$$

$$- \int_\Omega (\cancel{fu} + \lambda f v)\, dx - \Big\{ \mu/2 \int_\Omega \cancel{|\nabla u|^2} dx + \int_\Omega \cancel{fu}\, dx \Big\}$$

$$J(u+\lambda v) - J(u) = \mu/2 \int 2\lambda \nabla v^T \nabla u + \lambda^2 |\nabla v|^2 dx - \int \lambda f v\, dx$$

$$J(u+\lambda v) - J(u) = \mu/_2 \int_\Omega 2\lambda \nabla v^T \nabla u + \lambda^2 |\nabla v|^2 dx - \int_\Omega \lambda f v dx$$

optimality condn: $\lim_{\lambda \to 0} \dfrac{J(u+\lambda v) - J(u)}{\lambda} = 0$ for all $v \in V$

$$\frac{J(u+\lambda v) - J(u)}{\lambda} = \mu/_2 \int_\Omega 2 \nabla v^T \nabla u + \lambda |\nabla v|^2 dx - \int_\Omega f v dx$$

$\lim_{\lambda \to 0} \dfrac{J(u+\lambda v) - J(u)}{\lambda} = 0$ becomes

$$\mu/_2 \int 2 \nabla v^T \nabla u \, dx - \int_\Omega f v \, dx = 0$$

true for all $v \in V$.

$$\mu \int_\Omega \nabla u . \nabla v \, dx - \int_\Omega f v \, dx = 0$$

vector dot product

Opt. condn for u that minimizes $J(v) = \mu/_2 \int_\Omega |\nabla v|^2 dx - \int_\Omega f v dx$

is given by: $\lim_{\lambda \to 0} \dfrac{J(u+\lambda v) - J(u)}{\lambda} = 0 \quad \forall v \in V$

$$\mu \int_\Omega \nabla u . \nabla v \, dx - \int_\Omega f v \, dx = 0 \quad \forall v \in V$$

apply $\int_\Omega \nabla . \vec{F} \, dx = \int_{\partial\Omega} \hat{n} . \vec{F} \, ds$ (Divergence Thm)

$\forall v \in V$

apply $\int \cdots$ $\partial\Omega$

$$\hookrightarrow \mu \int_{\Omega} -\nabla\cdot\nabla u \, v \, dx - \int_{\Omega} f v \, dx = 0 \quad \forall v \in V$$

$$\int_{\Omega} \mu \Delta u \, v \, dx - \int_{\Omega} f v \, dx = 0 \quad \forall v \in V$$

in particular, $\int_{\Omega} (\mu \Delta u + f) v \, dx = 0$

$$\Rightarrow \int_{\Omega} (\mu \Delta u + f) v \, dx = 0 \quad \forall v \in V$$

$$\Rightarrow \mu \Delta u + f = 0 \text{ in } \Omega \text{ or}$$

$$\left\{ \begin{aligned} -\mu \Delta u &= f \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned} \right\}$$