Cps-5310

Tuesday, May 5, 2020 3:02 PM

Notation: $u(x_1,x_2) : \Omega \to \mathbb{R}$; $v : \Omega \to \mathbb{R}$

$$\nabla u = \begin{bmatrix} \partial_{x_1} u \\ \partial_{x_2} u \end{bmatrix}_{2\times 1} \quad \partial_{x_i} \equiv \partial_i \quad \nabla v = \begin{bmatrix} \partial_1 v \\ \partial_2 v \end{bmatrix}$$

$$\nabla v^T = [\partial_1 v \quad \partial_2 v]$$

$$\nabla u \cdot \nabla v = \nabla v^T \nabla u$$

$$= [\partial v \quad \partial_2 v] \begin{bmatrix} \partial u \\ \partial_2 u \end{bmatrix}$$

$$= \partial v \, \partial u + \partial_2 v \, \partial u$$

$$= \begin{pmatrix} \partial_1 u \\ \partial_2 u \end{pmatrix} \cdot \begin{pmatrix} \partial_1 v \\ \partial_2 v \end{pmatrix}$$ Dot product bet. ∇u & ∇v.

$\nabla \cdot \vec{F} = \partial_1 F_1 + \partial_2 F_2$ where $\vec{F} = (F_1, F_2)$ and $\partial_i \equiv \frac{\partial}{\partial x_i}$

$\nabla \cdot \nabla u = \partial^2 u / \partial x_1^2 + \partial^2 u / \partial x_2^2 = \Delta u$

Divergence Theorem formula:

$$\int_\Omega \nabla u \cdot \nabla v \, dx = -\int_\Omega \Delta u \, v \, dx + \int_{\partial\Omega} \frac{\partial u}{\partial n} v \, ds$$

How? Take $\vec{F}(x,y) = v \nabla u$

and apply $\int_\Omega \nabla \cdot \vec{F} \, dx = \int_{\partial\Omega} \hat{n} \cdot \vec{F} \, ds$

gives $\int_\Omega \nabla \cdot (v \nabla u) dx = \int_{\partial\Omega} \hat{n} \cdot (v \nabla u) ds$

$$\int_\Omega \nabla u \cdot \nabla v \, dx + \nabla \cdot \nabla u \, v \, dx = \int_{\partial\Omega} (\hat{n} \cdot \nabla u) v \, ds$$

$$\int \nabla u \cdot \nabla v \, dx + \int \Delta u \, v \, dx = \int (\hat{n} \cdot \nabla u) \, v \, ds$$

$$\int_{\Omega} \nabla u \cdot \nabla v \, dx + \int_{\Omega} \Delta u \, v \, dx = \int_{\partial\Omega} (\hat{n} \cdot \nabla u) \, v \, ds$$

$$\Rightarrow \boxed{\int_{\Omega} \nabla u \cdot \nabla v \, dx = -\int_{\Omega} \Delta u \, v \, dx + \int_{\partial\Omega} \hat{n} \cdot \nabla u \, v \, ds}$$