CPS-5310

Tuesday, April 28, 2020 6:22 PM

PDEs

ODEs: $\frac{dx}{dt} = F(t, x)$ $\quad x = (x_1, x_2, \dots x_n)$

$\frac{\partial x}{\partial x_i}$

Euler-Lagrange Equations

concentration difference

Energy = Tendency to interact + Tendency to seperate

energy conservation of the system.

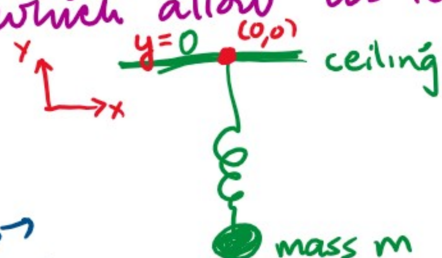
u → displacement

$E(u)$

$\boxed{E'(u^*) = 0}$ ⇒ $E''(u^*) > 0$ then u^* is displacement giving min energy

Euler-Lagrange Eqn

Task: Euler Lagrange eqns which allow us to derive ODEs & PDEs.



Hooke's Law

Energy $= \frac{K}{2}(y - y_0)^2$ $\quad y_0 \to$ rest length

positive

Gravitational energy $= -mgy$

Energy $E(y) = \boxed{\frac{K}{2}(y - y_0)^2 - mgy}$ (Potential energy)

calm position y^* which minimizes $E(y)$.

energy $E(y)$, $\frac{?}{2}$ ...

eqbm position y^* which minimizes $E(y)$.

$$E'(y) = 0 \rightarrow K(y^* - y_0) - mg = 0$$
$$y^* = \frac{mg + Ky_0}{K}$$

$$E''(y) = K > 0$$

for this system

$KE(y) = m\dot{y}^2/2$ $\quad v = \dot{y} = dy/dt$ velocity

Dynamics associated with spring system:

$$\frac{d}{dt}\left(\frac{\partial}{\partial v}(KE - \overset{\rightarrow E}{PE})\right) = \frac{\partial}{\partial y}(KE - PE)$$

$v = \dot{y} = dy/dt$ $\quad E = \frac{K}{2}(y-y_0)^2 - mgy$

$$KE - PE = m\frac{v^2}{2} - \left(\frac{K}{2}(y-y_0)^2 - mgy\right)$$

$$\frac{\partial}{\partial v}(\quad) = mv$$

$$\frac{d}{dt}\left(\frac{\partial}{\partial v}(KE-PE)\right) = \frac{d}{dt}(mv)$$

$$\frac{\partial}{\partial y}(KE-PE) = \frac{\partial}{\partial y}\left(m\frac{v^2}{2} - \left(\frac{K}{2}(y-y_0)^2 - mgy\right)\right)$$

$$= 0 - \frac{K}{2}2(y-y_0) + gm$$

Putting these two derivations together,

$LHS = \frac{d}{dt}(mv)$ $\quad RHS = gm - K(y-y_0)$

$$m\dot{v} = gm - K(y-y_0)$$
$$m\ddot{y} = mg - K(y-y_0) \quad \text{(Harmonic Oscillator)}$$

2D setting here for deriving Laplace Equation

$$\frac{\partial u}{\partial t} - \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0 \quad \text{(Heat Eq}^n\text{)}$$

2D setting here ...

$$\frac{\partial u}{\partial t} - \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = 0 \quad (\text{Heat Eq}^n)$$

$\frac{\partial u}{\partial x_i}$ partial deri wrt x_i

Δu Laplacian of $u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}$ = Trace of $D^2 u$

$D^2 u \downarrow$ Hessian of u

Heat equation

$$\underbrace{\frac{\partial u}{\partial t} - \Delta u = 0}_{} \rightarrow \frac{\partial u}{\partial t} = 0 \rightarrow \boxed{-\Delta u = 0}$$

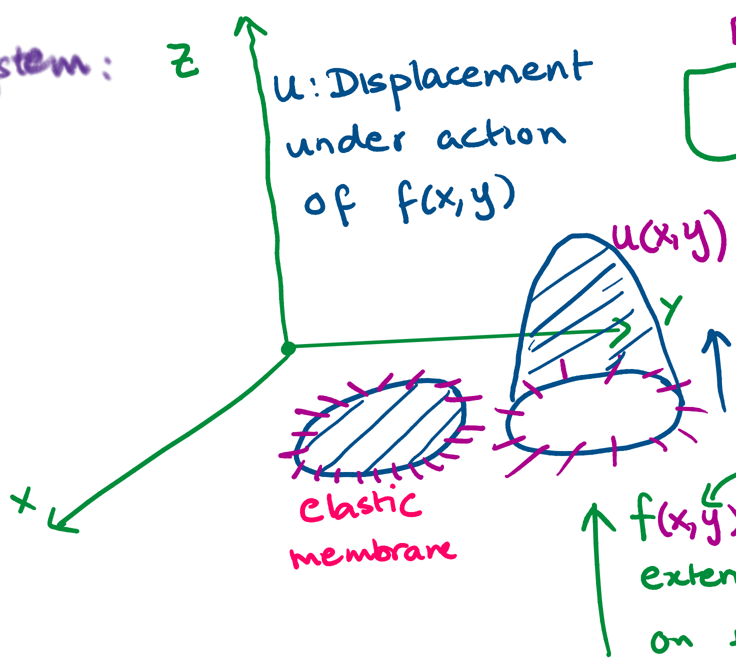
Laplace eq^n

$$\frac{u(t+h) - u(t)}{h} - \Delta u(t) = 0$$

$$\rightarrow -\Delta u(t) = F(u, t, t+h) \quad \text{(Numerical Time Derivative)}$$

Euler Lagrange equations for the system below* turn out to be $-\Delta u = f$

* System:



$E(u) =$

$\boxed{E'(u) = 0}$ give me Euler Lagrange eq^n $-\Delta u = f$

Δ: Laplacian

$\Omega \rightarrow$ circular membrane with boundary denoted by $\partial\Omega$.

$u = 0$ on $\partial\Omega$ because it is clamped on the boundary $\partial\Omega$

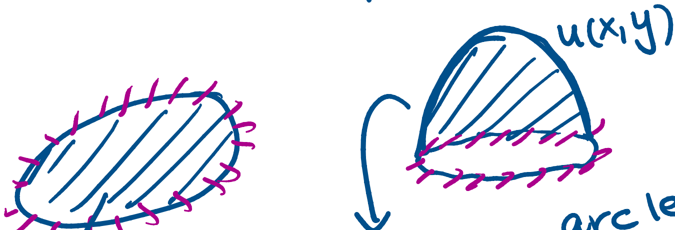
$u = 0$ on $\partial\Omega$ because it is clamped on the boundary $\partial\Omega$ (no displacement)

EQUILIBRIUM state $\rightarrow$ physical state where total energy is minimized.

Total energy

$$E(u) = PE(u) - J_f$$

$PE(u)$ = proportional to change in the surface area from state of rest



arc length of u

$$= \int_\Omega \sqrt{1 + (u')^2}\, dx - \int_\Omega 1\, dx$$

$$u' \rightarrow \begin{bmatrix} \partial u / \partial x \\ \partial u / \partial y \end{bmatrix} = \nabla u$$

$$PE(u) = \int_\Omega \sqrt{1 + |\nabla u|^2}\, dx - \int_\Omega 1\, dx$$

$|\nabla u| \ll 1 \rightarrow$ higher order terms can be neglected.

neglected.

$$PE(u) = \int_\Omega \underbrace{(1+|\nabla u|^2)^{1/2}}_{\substack{\text{Fractional} \\ \text{Binomial Thm}}} dx - \int_\Omega 1\, dx$$

Fractional Binomial Thm

$(1+|\nabla u|^2)^n = 1 + n|\nabla u|^2 + \dots$

$\simeq 1 + n|\nabla u|^2$

$\downarrow$

$1 + \frac{1}{2}|\nabla u|^2$

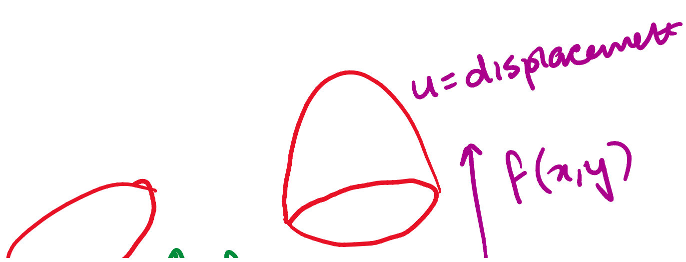
$$\simeq \int_\Omega \left(\left(1 + \frac{1}{2}|\nabla u|^2\right) - 1 \right) dx$$

$$PE(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 dx$$ $\longrightarrow$ small deflections u is small

$\Rightarrow$ $|\nabla u|^k$ neglected $k \geq 3$

$$\text{external force} = \int_\Omega f u \, dx$$

$$\text{Total energy } E(u) = \int_\Omega \frac{1}{2}|\nabla u|^2 dx - \int_\Omega f u \, dx$$



$\uparrow f$

$$E'(u) = 0 \longrightarrow -\Delta u = f$$

$\downarrow$

Gateaux Derivative $\frac{\delta E}{\delta v}$ of E / Directional Deri in the directn v.

$\Omega = [0,1]^2$

$f(x,y) = 2\pi^2 \sin \pi x \sin \pi y$