Error Analysis

Thursday, October 10, 2019 1:25 PM

$f(x) = e^x$ $\quad x = 0, 0.5, 1, -1$ $\quad n=3$

$\quad\quad\quad\quad\quad\quad x_0 \;\; x_1 \;\; x_2 \;\; x_3$

Formula:

$$f(x) - p_3(x) = \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{4!} f^{(4)}(c_x)$$

$f^{(4)}(c_x)$

fourth order derivative of e^{c_x}

Note: x lies between

min of $x_0, x_1, x_2, x_3 = -1$

and

max of $x_0, x_1, x_2, x_3 = 1$

$$-1 \leq x \leq 1$$

c_x also lies between -1 and 1.

$|f(x) - p_3(x)| \leq \; ?$ $\quad -1 \leq x \leq 1$

$$|f(x) - p_3(x)| = \left| \frac{x(x-0.5)(x-1)(x+1)}{24} \right| e^{c_x}$$

$c_x = 1$

$$\leq |x(x-0.5)(x-1)(x+1)| \frac{e^1}{24}$$

how to bound it?

Ans: Divide and conquer!

$|x| \leq 1$

$$|x(x-0.5)(x-1)(x+1)| = |x| \, |(x-0.5)(x-1)(x+1)|$$

$$\leq 1 \; |(x-0.5)(x-1)(x+1)|$$

$$= \underbrace{|x-0.5|}_{x=-1} \; |(x-1)(x+1)|$$

$$\leq 1.5 \; |x-1| \, |x+1|$$

$\leq 1.5 \quad |\ldots| \quad |\ldots|$

$x=-1 \qquad x=1$

$\leq 1.5 \quad |-2| \quad |2|$

$= 1.5 * 4 = 6$

$$|f(x) - p_3(x)| \leq 6 * \frac{e}{24} = 0.6795$$

Alternate way : $|x(x-0.5)(x-1)(x+1)| \leq ?$

$g(x) = x(x-0.5)(x-1)(x+1)$

apply Calculus to obtain the max. value

$g'(x^*) = 0 \Rightarrow x^*$ is extrema and check second order derivative...

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example 2: $f(x) = \sqrt{x} \qquad x = 1, 4, 9, 16$
$\qquad\qquad\qquad\qquad\qquad\qquad x_0 \; x_1 \; x_2 \; x_3$

Last Lecture: Constructed the lag & Newton's D.D poly of deg 3.

$$|f(x) - p_3(x)| = \left| \frac{(x-x_0)(x-x_1)(x-x_2)(x-x_3)}{4!} f^{(4)}(c_x) \right|$$

$$\leq ?$$

Where do x and c_x live?

$1 = \min x_i \leq x \leq \max x_i = 16$

$1 \leq c_x \leq 16$

What is $f^{(4)}(c_x)$?

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$f(x) = \sqrt{x}$ $\quad f'(x) = \frac{1}{2\sqrt{x}} = \frac{1}{2}x^{-1/2}$

$f''(x) = \frac{1}{2} * -\frac{1}{2} x^{-3/2}$ $\quad f'''(x) = \frac{1}{2} * -\frac{1}{2} * -\frac{3}{2} x^{-5/2}$

$$\boxed{f^{(4)}(x) = \frac{1}{2} * -\frac{1}{2} * -\frac{3}{2} * -\frac{5}{2} x^{-7/2} = -\frac{15}{16} x^{-7/2}}$$

$$|f(x) - P_3(x)| \le \frac{|(x-1)(x-4)(x-9)(x-16)|}{24} \underbrace{\left|\frac{-15}{16} c_x^{-7/2}\right|}_{f^{(4)}(c_x)}$$

$1 \le x \le 16$

$1 \le c_x \le 16$

$$= \frac{15|(x-1)(x-4)(x-9)(x-16)|}{24 * 16} \quad \frac{1}{c_x^{7/2}}$$

Once again,

$|(x-1)(x-4)(x-9)(x-16)| \le ?$

$\frac{1}{c_x^{7/2}} \le ?$ $\quad c_x = 1$

Please work out the details for these two bounds. Look at lecture notes for hints.

<u>Note</u>: $\underbrace{(x-1)(x-4)}\ \underbrace{(x-9)(x-16)}$

$|x^2 - 5x + 4|$ $\quad |x^2 - 25x + 144|$

$|x^2 - 5x + 4| \leq$ Bound $\underbrace{|x^3 - 2x + \dots|}_{\text{Bound}}$

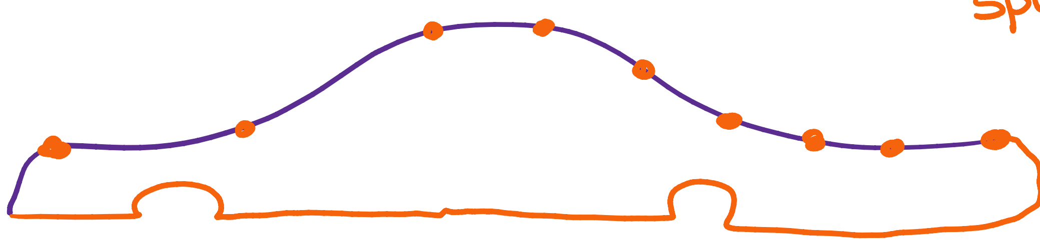
Splines: functions which also interpolate data.

These functions satisfy more properties than just regular Lagrange/Newton's D.D poly.

Most useful property:
splines are smooth functions interpolating data.

Why is this property desirable?

spline



Spline example

11/Section 4.3

$$S(x) = \begin{cases} -5 + 8x - 6x^2 + 2x^3 \; S_1(x) & 1 \leq x \leq 2, \\ 27 - 40x + 18x^2 - 2x^3 \to S_2(x) & 2 \leq x \leq 3. \end{cases}$$

Verify that $s(x)$ is a spline on $[1, 3]$.

Check if $S(x)$ is a natural cubic spline.

[i.e, $S''(x_0) = S''(x_2) = 0$?]

$x_0 \quad x_1 \quad x_2$

Property 1: $S(x)$ is a cubic poly on $[x_0, x_1]$ and $[x_1, x_2]$. True!

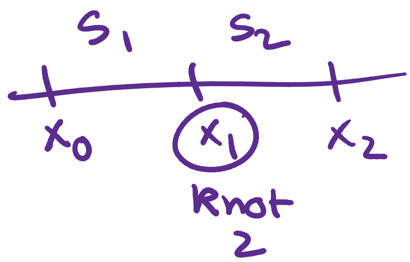
... Applicable ($S(x_i) = y_i$ no y_i value

$[x_0, x_1]$ and $[x_1, x_2]$. True!

Property 2: Not Applicable ($S(x_i) = y_i$ no y_i value is provided)

Property 3: checking

$$\lim_{x \to x_1^-} S'(x) = \lim_{x \to x_1^+} S'(x)$$



$$\lim_{x \to 2^-} S'(x) = \lim_{x \to 2^+} S'(x)$$

$$\lim_{x \to 2^-} S_1'(x) = \lim_{x \to 2^+} S_2'(x)$$

$S_1(x) = -5 + 8x - 6x^2 + 2x^3$

$S_1'(x) = 8 - 12x + 6x^2 \qquad S_1'(2) = 8$

$S_2'(x) = -40 + 36x - 6x^2 \qquad S_2'(2) = 8$

so property 3 holds!

Prop 4: $\lim_{x \to 2} S_1''(x) = \lim_{x \to 2} S_2''(x)$

$S_1'(x) = 8 - 12x + 6x^2 \qquad S_2'(x) = -40 + 36x - 6x^2$

$S_1''(x) = -12 + 12x \qquad S_2''(x) = 36 - 12x$

$S_1''(2) = 12 \qquad S_2''(2) = 36 - 24 = 12$

$S(x)$ satisfies property 4 as well!

$\Rightarrow$ $S(x)$ is a spline.

check if $S(x)$ is a cubic spline ?

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$s''(1) = 0$?

$s_1''(x) = -12+12x$

$s_1''(1) = 0$

True

$s''(3) = 0$?

$s_2''(x) = 36-12x$

$s_2''(3) = 0$ True

Since the above is true $S(x)$ is a Natural cubic Spline!

#14

$$S(x) = \begin{cases} x^3 & s_1(x) & 0 \le x \le 1 \\ 2x-1 & s_2(x) & 1 < x < \boxed{2} \\ 3x^2-9 & s_3(x) & \boxed{2} \le x \le 3 \end{cases}$$

Prop1, Prop2 hold true!

check if prop 3 & 4 are satisfied!

$$\lim_{x \to 1} s_1'(x) \stackrel{?}{=} \lim_{x \to 1} s_2'(x)$$

$$s_2'(x) = 2$$

$$s_1'(x) = 3x^2$$

$$\neq$$

$$\lim_{x \to 1} s_1'(x) = 3$$

$$\lim_{x \to 1^-} s'(x) = 3 \neq 2 = \lim_{x \to 1^+} s'(x)$$

Not a spline!