Thursday, October 3, 2019 2:02 PM

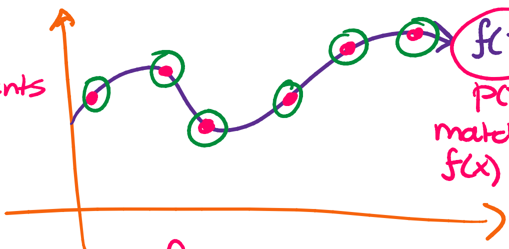
$y_i = f(x_i)$ complicated function

other application (Taylor poly) $(x_i, f(x_i))$

Taylor poly

$f(0) = p(0)$

p(x) represents f(x)



f(x)

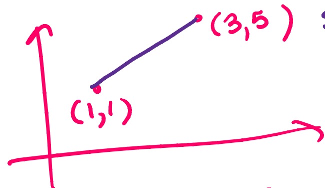
p(x) matching f(x) at x_0, x_1, x_2, ⋮

$$\begin{cases} p(x_0) = f(x_0) \\ p(x_1) = f(x_1) \\ \vdots \end{cases}$$

$\int f(x)\,dx \approx \int p(x)\,dx$ (easy to calculate)

Main Ques: How to construct these polynomials?

start simple (x_0, y_0) & (x_1, y_1)



straight line $p_1(x)$ interpolates 2 data points.

$$p_1(x) = \overset{1}{y_0}\, l_0(x) + \overset{5}{y_1}\, l_1(x)$$

where $l_0(x) = \frac{(x_1 - x)}{x_1 - x_0}$ and $l_1(x) = \frac{x - x_0}{x_1 - x_0}$

check if (1,1) and (3,5) are interpolated by $p_1(x)$?

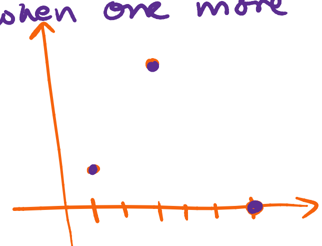
$$p_1(x) = 1 * \frac{(3-x)}{3-1} + 5 * \frac{(x-1)}{3-1} = \frac{3-x}{2} + \frac{5}{2}(x-1)$$

$$p_1(1) = \frac{3-1}{2} + 0 = 1 \quad \text{passes through } (1,1)$$

$$p_1(3) = 0 + \frac{5(3-1)}{3-1} = 5 ✓ \quad \text{passes through } (3,5).$$

I want to derive the formula when one more date point is added: $\underbrace{(6,0)}_{(x_2, y_2)}$

(1,1), (3,5)



$$\boxed{\begin{matrix} x_0 = 1 & y_0 = 1 \\ x_1 = 3 & y_1 = 5 \\ x_2 = 6 & y_2 = 0 \end{matrix}}$$

3 points ⇒ poly of deg 2 to represent the 3 condns.

I want $p_2(x) = ax^2 + bx + c$ to satisfy:

$$\left.\begin{matrix} p_2(1) = 1 \\ p_2(3) = 5 \\ p_3(\underline{6}) = 0 \end{matrix}\right\} \Rightarrow \begin{bmatrix} 1 & 1 & 1 \\ 9 & 3 & 1 \\ 36 & 6 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 5 \\ 0 \end{bmatrix}$$

$$A \qquad \mathbf{x} \qquad = b$$

$P_3(6) = 0$ ... $A \quad x \quad = b$

$$\boxed{P_1(x) = y_0 l_0(x) + y_1 l_1(x)}$$

2 data points (x_0, y_0) (x_1, y_1)

I want $p_2(x) = y_0 l_0(x) + y_1 l_1(x) + y_2 l_2(x)$

$l_0(x_0) = 1, \quad l_0(x_1) = l_0(x_2) = 0$ → $(x-x_1)(x-x_2)$

$l_1(x_0) = 0 \quad l_1(x_1) = 1 \quad l_1(x_2) = 0$ → $(x-x_0)(x-x_2)$

$l_2(x_0) = 0 \quad l_2(x_1) = 0 \quad l_2(x_2) = 1$ → $(x-x_0)(x-x_1)$

$$l_0(x_0) = \frac{(x_0-x_1)(x_0-x_2)}{(x_0-x_1)\ (x_0-x_2)} = 1 \qquad p_2(x_0) = y_0 l_0(x_0)$$

$$\boxed{l_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}} \Rightarrow l_0(x_0) = \frac{(\cancel{x_0-x_1})(\cancel{x_0-x_2})}{(\cancel{x_0-x_1})(\cancel{x_0-x_2})}$$

$$\boxed{l_0(x) = (x-x_1)(x-x_2)} \rightarrow l_0(x_0) = (x_0-x_1)(x_0-x_2)$$

$l_0(x_1) = 0 = l_0(x_2)$

but $l_0(x_0) \neq 1$ → how much am I off by $(x_0-x_1)(x_0-x_2)$

$$l_0(x_0) = \frac{1}{(x_0-x_1)(x_0-x_2)}$$

$$l_0(x) = \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}$$

$$l_0(x_0) = \frac{(x_0-x_1)(x_0-x_2)}{(x_0-x_1)(x_0-x_2)} = 1$$

$l_1(x)$

$\underbrace{l_1(x_0) = 0}_{x-x_0}$ $\quad l_1(x_2) = 0$ but $l_1(x_1) = 1$

$$l_1(x) = (x-x_0)\,(x-x_2)$$

violates $l_1(x_1) = 1$

$$l_1(x_1) = (x_1-x_0)\,(x_1-x_2)$$

$l_1(x_1)$ is $(x_1-x_0)(x_1-x_2)$ instead of 1

$$l_1(x) = \frac{(x-x_0)\,(x-x_2)}{(x_1-x_0)\,(x_1-x_2)}$$

$l_1(x_0) = l_1(x_2) = 0$

$$l_1(x_1) = \frac{(\cancel{x_1-x_0})\,(\cancel{x_1-x_2})}{(\cancel{x_1-x_0})\,(\cancel{x_1-x_2})}$$

$l_2(x)$ $\quad l_2(x_0) = 0 \quad (x-x_0)$

$\quad l_2(x_1) = 0 \quad (x-x_1)$

$$l_2(x) = (x-x_0)\,(x-x_1)$$

$l_2(x_0) = 0$ ✓

$l_2(x_1) = 0$ ✓

but $l_2(x_2) = \underline{(x_2-x_0)\,(x_2-x_1)}$

$$l_2(x) = \frac{(x-x_0)\,(x-x_1)}{\underbrace{(x_2-x_0)\,(x_2-x_1)}}$$

scaling factor

General Formula:

$$\boxed{p_2(x) = y_0 l_0(x) + y_1 l_1(x) + y_2 l_2(x)}$$

$p_2(x_0)$ $= y_0 * l_0(x_0) + y_1 * 0 + y_2 * 0 = y_0 * 1 = y_0$

$$P_2(x_0) = y_0 * l_0(x_0) + y_1 * 0 + y_2 * 0 = y_0 * 1 = y_0$$

$$l_1(x_0) = 0 \qquad l_2(x_0) = 0$$

$$P_2(x_1) = y_0 * 0 + y_1 * 1 + y_2 * 0 = y_1$$

$$P_2(x_2) = y_0 * 0 + y_1 * 0 + y_2 * 1 = y_2$$

Original Ques: $\underline{x_0} = 1 \quad y_0 = 1$, $\;(3, 5)\;$ $(6, 0)$

$\qquad\qquad\qquad\qquad\qquad\qquad\quad \underline{x_1}\ y_1 \qquad x_2\ y_2$

$$P_2(x) = y_0 * \frac{(x-3)(x-6)}{(1-3)(1-6)} + y_1 \frac{(x-1)(x-6)}{(3-1)(3-6)} + y_2 \frac{(x-1)(x-3)}{(6-1)(6-3)}$$

$$x_0 = 1 \qquad\qquad x_1 = 3 \qquad\qquad x_2 = 6$$

$$P_2(x) = \frac{(x-3)(x-6)}{(-2)(-5)} + 5 \frac{(x-1)(x-6)}{(2 * -3)} + 0$$

check

$$P_2(1) = \frac{(1-3)(1-6)}{(-2) * (-5)} = 1$$

$$P_2(3) = 5 \frac{(\cancel{3-1})(\cancel{3-6})}{\cancel{2} * \cancel{-3}} = 5$$

$$P_2(6) = 0 + 0 = 0$$

$P_2(x)$ passes through $(1, 1)$, $(3, 5)$ & $(6, 0)$.

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4 data points : $(x_0, y_0), (x_1, y_1), (x_2, y_2)$ & (x_3, y_3)

$$P_3(x) = y_0 l_0(x) + y_1 l_1(x) + y_2 l_2(x) + y_3 l_3(x)$$

$l_0(x) \Rightarrow l_0(x_0) = 1$ but $\underline{l_0(x_i) = 0}$ $\; i = 1, 2, 3$

$$l_0(x_1) = 0 \Rightarrow (x - x_1)(x - x_2)(x - x_3)$$

$$l_0(x_2) = 0$$

$$l_0(x_3) = 0$$

$$l_0(x) = (x - x_1)(x - x_2)(x - x_3)$$

$$l_0(x) = (x-x_1)(x-x_2)(x-x_3)$$

$$l_0(x_1) = l_0(x_2) = l_0(x_3) = 0 \quad \text{but}$$

$$l_0(x_0) = (x_0-x_1)(x_0-x_2)(x_0-x_3)$$

$$l_0(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}$$

Similarly for $l_1(x)$ & $l_2(x)$ & $l_3(x)$.