Math 4329 Iterative methods: Jacobi & Gauss Seidel Method

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$$2x_1 - 6x_2 = 3$$
$$x_1 + x_2 = 5$$

apply 2 iterations of Jacobi & Gauss Seidel Method given initial guess $x^{(0)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

iteration #

Jacobi method:

$$x_1^{(1)} \leftarrow 2x_1 - 6x_2 = 3 \rightarrow (1)$$
$$x_2^{(1)} \leftarrow x_1 + x_2 = 5 \rightarrow (2)$$

$$x^{(0)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \longrightarrow x^{(1)} = \begin{pmatrix} x_1^{(1)} \\ x_2^{(1)} \end{pmatrix} = \begin{bmatrix} 3/2 \\ 5 \end{bmatrix}$$

$x_1^{(0)} = 0 \quad x_2^{(0)} = 0$

use eq^n 1 $\quad 2x_1^{(1)} - 6x_2^{(0)} = 3$

$2x_1^{(1)} = 3 \Rightarrow x_1^{(1)} = 3/2$

use eq^n 2 to update $x_2^{(1)}$ given $x_1^{(0)} = 0$

$\underset{0}{x_1^{(0)}} + x_2^{(1)} = 5 \rightarrow x_2^{(1)} = 5$

$$x^{(1)} = \begin{bmatrix} 3/2 \\ 5 \end{bmatrix} \longrightarrow x^{(2)} = \begin{bmatrix} x_1^{(2)} \\ x_2^{(2)} \end{bmatrix} = \begin{bmatrix} 33/2 \\ 7/2 \end{bmatrix}$$

$x_2^{(1)}$

eq^n 1: $\quad 2x_1^{(2)} - 6x_2^{(1)} = 3$

eq^n 2: $\quad x_1^{(1)} + x_2^{(2)} = 5$

$2x_1^{(2)} - 6(5) = 3 \rightarrow 2x_1^{(2)} = 3 + 30 \quad x_1^{(2)} = \frac{33}{2}$

$x_1^{(1)} + x_2^{(2)} = 5 \rightarrow 3/2 + x_2^{(2)} = 5$

$x_2^{(2)} = 5 - 3/2 = 7/2$

Note:

$$\begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix} \begin{bmatrix} x_1^{(2)} \\ x_2^{(2)} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} - \begin{bmatrix} 0 & -a_{12} \\ -a_{22} & 0 \end{bmatrix} \begin{bmatrix} x_1^{(1)} \\ x_2^{(1)} \end{bmatrix}$$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \quad N = \begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix} \quad P = \begin{bmatrix} 0 & -a_{12} \\ -a_{22} & 0 \end{bmatrix}$$

then Jacobi method: $\quad N x^{(2)} = b + P x^{(1)}$

$$N x^{(3)} = b + P x^{(2)}$$
$$N x^{(4)} = b + P x^{(3)}$$

This is the matrix-vector formula for Jacobi method

$$Ax = b$$

$$N x^{(1)} = b + P x^{(0)}$$

$$A = \begin{bmatrix} 2 & -6 \\ 1 & 1 \end{bmatrix} \quad x = ? \quad b = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$x^{(0)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \rightarrow x^{(1)} \text{ use } N x^{(1)} = b + P x^{(0)}$$

$$N = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \qquad P = \begin{bmatrix} 0 & 6 \\ -1 & 0 \end{bmatrix} \quad P = \begin{bmatrix} 0 & -a_{12} \\ -a_{21} & 0 \end{bmatrix}$$

$$b = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

$$N \; x^{(2)} = b + P x^{(1)}$$

Recall N^{-1} denotes inverse of N. $\left(N * N^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right.$, $\left. N^{-1} * N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$

$$N \; x^{(1)} = b + P x^{(0)}$$

$$N^{-1} N \; x^{(1)} = N^{-1}(b + P x^{(0)}) \rightarrow x^{(1)} = N^{-1} b + N^{-1} P x^{(0)}$$

$$x^{(2)} = N^{-1} b + N^{-1} P x^{(1)}$$

$\vdots$

Gauss Seidel 2 iterations based on $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ as initial guess.

1 eq^n to update x_1 $\quad 2x_1 - 6x_2 = 3 \rightarrow (1)$

2 eq^n to update x_2 $\quad x_1 + x_2 = 5 \rightarrow (2)$

$$x^{(0)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad x^{(1)} = \begin{pmatrix} 3/2 \\ 7/2 \end{pmatrix} \qquad x^{(2)} = \begin{bmatrix} \; \\ \; \end{bmatrix}$$

$$2x_1^{(1)} - 6x_2^{(0)} = 3 \rightarrow x_1^{(1)} = 3/2$$

$$x_1^{(1)} + x_2^{(1)} = 5$$

$$x^{(1)} = 5 \rightarrow x_2^{(1)} = 5 - 3/2 = 7/2$$

$$x_1^{(1)} + x_2^{(1)} = 5$$

$$\frac{3}{2} + x_2^{(1)} = 5 \Rightarrow x_2^{(1)} = 5 - \frac{3}{2} = \frac{7}{2}$$

$$x^{(1)} = \begin{bmatrix} 3/2 \\ 7/2 \end{bmatrix} \longrightarrow x^{(2)} = \begin{bmatrix} 12 \\ -7 \end{bmatrix}$$

$$x_1^{(2)} \rightarrow \quad 2x_1^{(2)} - 6x_2^{(1)} = 3$$

$$2x_1^{(2)} - 6 * 7/2 = 3 \rightarrow 2x_1^{(2)} = 21 + 3$$

$$x_1^{(2)} = 12$$

$$x_2^{(2)} \rightarrow \quad x_1^{(2)} + x_2^{(2)} = 5$$

$$12 + x_2^{(2)} = 5 \rightarrow \underline{x_2^{(2)} = -7}$$

Gauss Seidel formula using matrix vector form:

$$N x^{(1)} = b + P x^{(0)}$$

$$N x^{(2)} = b + P x^{(1)}$$

$$\vdots$$

$$b = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \qquad A = \begin{bmatrix} 2 & -6 \\ 1 & 1 \end{bmatrix}$$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad N = \begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{bmatrix}, \quad P = N - A = \begin{bmatrix} 0 & -a_{12} \\ 0 & 0 \end{bmatrix}$$

$$N x^{(1)} = b + P x^{(0)}$$

$$N x^{(2)} = b + P x^{(1)}$$

$$A = \begin{bmatrix} 2 & -6 \\ 1 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 3 \\ 5 \end{bmatrix}$$

for above problem,

$$N = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$$

$$P = \begin{bmatrix} 0 & 6 \\ 0 & 0 \end{bmatrix}$$

$$x^{(0)} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad N x^{(1)} = b + P x^{(0)}$$

$$N^{-1} b + N^{-1} P x^{(0)}$$

$x^{(0)} = \begin{pmatrix} 0 \end{pmatrix}$ $\quad N x^{(1)} = b + P x^{(0)}$

$$N^{-1} N x^{(1)} = N^{-1} b + N^{-1} P x^{(0)}$$

$$x^{(1)} = N^{-1} b + N^{-1} P x^{(0)}$$

$$x^{(2)} = N^{-1} b + N^{-1} P x^{(1)}$$

N^{-1} = inverse of N $\quad N^{-1} N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = N N^{-1}$

CONVERGENCE ANALYSIS

$x^{(0)} \to x^{(1)} \to x^{(2)} \text{ - - -}$

$$N x^{(k+1)} = b + P x^{(k)} \qquad k = 0, 1, 2, \dots$$

which solves $Ax = b$. $\quad x^{(0)}$ initial Guess

$\{x^{(k)}\}_{k \geq 1}$ based on $\boxed{N x^{(k+1)} = b + P x^{(k)} \ \& \ x^{(0)}.}$

Assume that x^* solves $Ax = b$

i.e $Ax^* = b$. $\quad$ Assume $x^{(k)} \to x^*$ as $k \to \infty$.

Consider ① $\boxed{N x^{(k+1)} = b + P x^{(k)}}$ $\quad k = 1, 2, \dots$

$$\lim_{k \to \infty} N x^{(k+1)} = b + \lim_{k \to \infty} P x^{(k)}$$

② $\boxed{N x^* = b + P x^* \qquad \text{since } \lim_{k \to \infty} x^k = x^*}$

② - ①

$$N x^* - N x^{(k+1)} = b + P x^* - b - P x^{(k)}$$

$$N (x^* - x^{(k+1)}) = P (x^* - x^{(k)})$$

$k = 0$ & $e^{(1)} = x^* - x^{(1)}$

$e^{(0)} = x^* - x^{(0)}$

$$N^{-1} N e^{(1)} = N^{-1} P e^{(0)} \to e^{(1)} = N^{-1} P e^{(0)}$$

$$N e^{(2)} = P e^{(1)} \to e^{(2)} = N^{-1} P e^{(1)}$$

$$N e^{(3)} = P e^{(2)} \to e^{(3)} = N^{-1} P e^{(2)}$$

$e^{(1)} = N^{-1}Pe^{(0)}$

$e^{(2)} = N^{-1}P\, e^{(1)} = N^{-1}P\left(N^{-1}Pe^{(0)}\right) = (N^{-1}P)^2 e^{(0)}$

$\vdots$

$e^{(k+1)} = (N^{-1}P)^{k+1} e^{(0)}$

use matrix-vector property: $\|Az\| \leq \|A\|\,\|z\|$

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A matrix z vector

$e^{(1)} = x^* - x^{(1)} = N^{-1}P\, e^{(0)}$

$\|e^{(1)}\| = \|\underbrace{N^{-1}P}_{\text{matrix}}\, \underbrace{e^{(0)}}_{\text{vector}}\| \leq \|N^{-1}P\|\;\|e^{(0)}\|$

$\|e^{(2)}\| = \|N^{-1}Pe^{(1)}\| \leq \|N^{-1}P\|\;\|e^{(1)}\|$

$\leq \|N^{-1}P\|\left(\|N^{-1}P\|\;\|e^{(0)}\|\right)$

$= \|N^{-1}P\|^2\;\|e^{(0)}\|$

$$\boxed{\begin{array}{l}\text{here if } \|N^{-1}P\| < 1 \text{ then} \\ N\,x^{(k+1)} = b + P\,x^{(k)} \quad k = 0, 1, \ldots \\ \text{converges for any } x^{(0)}.\end{array}}$$

$$A = \begin{bmatrix} 2 & -6 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Jacobi: $N x^{(k+1)} = b + Px^{(k)} \quad k = 0, 1, \ldots$

$$N = \begin{bmatrix} a_{11} & 0 \\ 0 & a_{22} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$

$N = \begin{bmatrix} 0 & a_{22} \end{bmatrix}^{-1} \begin{bmatrix} 0 & 1 \end{bmatrix}$

$P = \begin{bmatrix} 0 & -a_{12} \\ -a_{22} & 0 \end{bmatrix} = \begin{bmatrix} 0 & 6 \\ -1 & 0 \end{bmatrix}$

$\|N^{-1}P\| = ?$

<u>Note</u>: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $A^{-1} = \frac{1}{\det A}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, $\det A = ad - bc$

$N = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ $N^{-1} = \frac{1}{\det N}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ $\det N = 2 - 0 = 2$

$= \begin{pmatrix} 1/2 & 0 \\ 0 & 1 \end{pmatrix}$

$$N^{-1}P = \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 & 6 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 6/2 \\ -1 & 0 \end{bmatrix} \quad \begin{array}{c} \text{Row sum} \\ 6/2 \\ 1 \end{array}$$

$\|N^{-1}P\| = 6/2 = 3 > 1 \Rightarrow$ Jacobi method doesnot converge!

<u>Gauss Seidel</u>:

$N x^{(k+1)} = b + P x^{(k)}$, $k = 0, 1, \ldots$

$A = \begin{bmatrix} 2 & -6 \\ 1 & 1 \end{bmatrix}$ $N = \begin{bmatrix} a_{11} & 0 \\ a_{21} & a_{22} \end{bmatrix}$ $P = \begin{bmatrix} 0 & -a_{12} \\ 0 & 0 \end{bmatrix}$

$P = \begin{bmatrix} 0 & 6 \\ 0 & 0 \end{bmatrix}$

$N = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$

$N^{-1} = \frac{1}{\det N}\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1/2 & 0 \\ -1/2 & 1 \end{bmatrix}$

rowsum

$\det N = 2$

$$N^{-1}P = \begin{bmatrix} 1/2 & 0 \\ -1/2 & 1 \end{bmatrix} \begin{bmatrix} 0 & 6 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 3 \\ 0 & -3 \end{bmatrix} \begin{array}{l} 0+3 \\ 0+3 \end{array}$$

$\|N^{-1}P\| = 3 > 1 \to$ no convergence

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for the system $\begin{cases} 4x_1 - x_2 = 1 \\ x_1 - 4x_2 = 0 \end{cases}$

determine (without calculating any iterations) whether Jacobi or Gauss Seidel will converge.

$$A = \begin{bmatrix} 4 & -1 \\ 1 & -4 \end{bmatrix} \qquad Nx^{(k+1)} = b + Px^{(k)}$$

Jacobi: $N = \begin{bmatrix} 4 & 0 \\ 0 & -4 \end{bmatrix} \quad N^{-1} = \begin{bmatrix} 1/4 & 0 \\ 0 & -1/4 \end{bmatrix}$

$$P = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$N^{-1}P = \begin{bmatrix} 1/4 & 0 \\ 0 & -1/4 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1/4 \\ 1/4 & 0 \end{bmatrix} \begin{array}{l} \text{Row Sum} \\ 0+1/4 \\ 1/4+0 \end{array}$$

$\|N^{-1}P\| = 1/4 < 1 \Rightarrow$ convergence!

Gauss Seidel

$$A = \begin{bmatrix} 4 & -1 \\ 1 & -4 \end{bmatrix} \qquad Nx^{(k+1)} = b + Px^{(k)}$$

$$A = \begin{bmatrix} & \\ 1 & -4 \end{bmatrix}$$

$$N = \begin{bmatrix} 4 & 0 \\ 1 & -4 \end{bmatrix} \qquad P = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = N - A$$

$\|N^{-1}P\| < 1$?

$$N = \begin{bmatrix} 4 & 0 \\ 1 & -4 \end{bmatrix} \qquad \det N = -16 - 0 = -16$$

$$N^{-1} = -\frac{1}{16}\begin{bmatrix} -4 & 0 \\ -1 & 4 \end{bmatrix} = \begin{bmatrix} 4/16 & 0 \\ 1/16 & -4/16 \end{bmatrix} = \begin{bmatrix} 1/4 & 0 \\ 1/16 & -1/4 \end{bmatrix}$$

$$N^{-1}P = \begin{bmatrix} 1/4 & 0 \\ 1/16 & -1/4 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1/4 \\ 0 & 1/16 \end{bmatrix}$$

$\|N^{-1}P\| = 1/4 < 1 \Rightarrow$ convergence!

Remark:

$$\begin{cases} \underbrace{4}_{a_{11}} x_1 - x_2 = 1 & \rightarrow \text{eq}^n\ 1 \\ x_1 - \underbrace{4}_{a_{22}} x_2 = 0 & \rightarrow \text{eq}^n\ 2 \end{cases}$$

If I switch the eq^{n}s and consider the system of eq^{n}s:

$$x_1 - 4x_2 = 0 \rightarrow \text{eq}^n\ ①$$
$$4x_1 - x_2 = 1 \rightarrow \text{eq}^n\ ②$$

... convergence anymore!

I donot have convergence anymore!

$$A = \begin{bmatrix} 1 & -4 \\ 4 & -1 \end{bmatrix}$$

Diagonally Dominant

$|a_{11}| > |a_{12}|$

$|a_{22}| > |a_{21}|$