Online-math4329

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Goal → math Ques answer numerically.

↳ Iterative method

Last class find w_1 & w_2

$$w_1 f(0) + w_2 f(1) \approx \int_0^3 f(x)\,dx$$

$f(x) = 1 \rightarrow \quad w_1 + w_2 = 3$

$f(x) = x \rightarrow \quad w_2 = 9/2$

Solve linear system of 2 equations in unknowns w_1 & w_2.

① ← $x + y = 3$

② ← $y = 9/2$

⇒ 2 planes pts of intersectn

write ①-② in matrix-vector form.

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 9/2 \end{bmatrix}$$

check

$$\begin{bmatrix} 1*x + 1*y \\ 0*x + 1*y \end{bmatrix} = \begin{bmatrix} 3 \\ 9/2 \end{bmatrix}$$

$$A\vec{x} = b$$

$\vec{x} = \begin{pmatrix} x \\ y \end{pmatrix}$ $\quad A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ $\quad b = \begin{bmatrix} 3 \\ 9/2 \end{bmatrix}$

$Ax = b$

A: 2x2 matrix (rows x coloumns)

x: 2 rows 1 coloumn

Suppose I want to solve

$$Ax = b$$

A: 10^{10} rows x 10^{10} coloumns

vectors (x, b): 10^{10} rows * 1 coloumn

Need for Iteration methods.

Notation:

Review solving $Ax=b$ using $[A \,\vdots\, b]$ and backward subsitution.

$A \to$ Quality of a matrix (unstable)

Notation $n \to$ #rows = #cols

$n=2$

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2\times 2}$$

Problem: Suppose I want to construct a linear poly $p(x) = ax+b$

interpolating $(1,3)$ & $(0,2)$. without using Lagrange/NewtonDD formula.

$p(x) = ax+b$

$p(1) = 3 \to a*1 + b = 3$

$p(0) = 2 \qquad a*0 + b = 2$

$p(0) = 2$ $\qquad a * 0 + b = 2$

Linear system $\rightarrow$ $\boxed{\begin{aligned} a + b &= 3 \\ b &= 2 \end{aligned}}$

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad B = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

$$X = \begin{bmatrix} a \\ b \end{bmatrix} \qquad \boxed{AX = B}$$

$a = 1$ $\quad b = 2$ Answer

Backward Substitution Method & LU Decomposition

Solve the foll. sys of equations:

$$2x_1 + 3x_2 = 1 \rightarrow (1)$$

$$4x_1 + 7x_2 = 10 \rightarrow (2)$$

Step I: express (1) & (2) in $Ax = b$ format.

$$\begin{array}{c} R_1 \\ R_2 \end{array} \begin{bmatrix} 2_{a_{11}} & 3_{a_{12}} \\ 4_{a_{21}} & 7_{a_{22}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 10 \end{bmatrix}$$

$A \quad X \quad b$

Step 2: Goal Turn entry $a_{21} = 0$ using Row operation

2nd row

Step 2: Goal ...

2nd row
1st col.

$$R_2 \longrightarrow R_2 - \frac{a_{21}}{a_{11}} R_1 \,, \quad m_{21} = a_{21}/a_{11}$$

multiplier for Row 2 col 1.

$$m_{21} = \frac{a_{21}}{a_{11}} = \frac{4}{2} = 2$$

$$\begin{array}{c} R_1 \\ R_2 \end{array} \left[\begin{array}{cc|c} 2 & 3 & 1 \\ 4 & 7 & 10 \end{array}\right] \qquad [A \,|\, b]$$

$$R_2 \rightarrow R_2 - 2R_1$$

$$\begin{array}{c} R_1 \\ R_2 \end{array} \left[\begin{array}{cc|c} 2 & 3 & 1 \\ \underset{0}{4-2(2)} & \underset{1}{7-2(3)} & \underset{8}{10-2*1} \end{array}\right]$$

$$\left[\begin{array}{cc|c} 2 & 3 & 1 \\ 0 & 1 & 8 \end{array}\right]$$

$$\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 8 \end{bmatrix}$$

Backward Substitution:

$$2x_1 + 3x_2 = 1$$

$$x_2 = 8$$

start with x_2 then get x_1's valve

↓ start with x_2 then get x_1's value

by substituting $x_2 = 8$ into $2x_1 + 3x_2 = 1$.

$x_2 = 8$ in $2x_1 + 3x_2 = 1$ gives

$2x_1 + 3*8 = 1 \rightarrow 2x_1 = 1 - 24$

$x_1 = -\frac{23}{2}$

Obtained $\boxed{x_1 = -23/2 \ \& \ x_2 = 8}$ by Backward Substitution.

Why this strict Row operation?

$R_2 \rightarrow R_2 - m_{21} R_1$, m_{21} = mult. for Row2 col 1

$= \frac{a_{21}}{a_{11}}$

$$U = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}$$ special matrix entries below diagonal = 0

Upper Triangular

$$L = \begin{bmatrix} 1 & 0 \\ m_{21} & 1 \end{bmatrix}$$ special matrix entries above diagonal = 0

$$= \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$$

same!

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 7 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}}_{U}$$

L U

Decomposition

$A = LU$ then, solve $Ax = b$ by solving 2 simpler problems.

Afua's ~~Ques~~: $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$

solve $Ax = \begin{bmatrix} -3 \\ 4 \end{bmatrix}$ $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$

$$\boxed{\begin{aligned} x_1 + 0x_2 &= 3 \\ 2x_1 + x_2 &= 4 \end{aligned}}$$

Forward Substitution

$$\boxed{x_1 = 3}$$
$$2x_1 + x_2 = 4$$

$x_1 = 3$

Substitute $x_1 = 3$ into $2x_1 + x_2 = 4$

$6 + x_2 = 4 \Rightarrow x_2 = -2$

solve $Ax = b$ by ① expressing $A = LU$

where $L = \begin{bmatrix} 1 & 0 \\ m_{21} & 1 \end{bmatrix}$ $m_{21} = \dfrac{a_{21}}{a_{11}}$ and

$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$U = \begin{bmatrix} a_{11} & a_{12} \\ 0 & U_{22} \end{bmatrix}$,

$U_{22} = a_{22} - \dfrac{a_{21}}{} a_{12}$

$$U_{22} = a_{22} - \frac{a_{21}}{a_{11}} a_{12}$$

(2) & solving

$$Ax = b \xrightarrow{A = LU} LUx = b$$

$$\underbrace{Ux}_{y}$$

Solve 2 simpler problems:

$$Ly = b$$

L → lower triangular
FORWARD SUBSTITUTION

Once we have y,

solve

$$Ux = y$$

Upper triangular solved by BACKWARD SUBSTITUTION

Solve the following system* using the LU-decomposition of A and using forward and Backward substitution.

$$\begin{array}{c} R_1 \\ R_2 \end{array} \underbrace{\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} 2 \\ 5 \end{bmatrix}}_{b}$$

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \xrightarrow[m_{21} = \frac{a_{21}}{a_{11}}]{R_2 \to R_2 - m_{21} R_1} \overset{U}{\begin{bmatrix} 1 & 2 \\ 0 & U_{22} \end{bmatrix}}$$

$$A = \begin{bmatrix} 1 & 0 \\ m_{21} & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 0 & U_{22} \end{bmatrix}$$

$$\begin{array}{cc} \begin{bmatrix} m_{21} & 1 \end{bmatrix} & \begin{bmatrix} 0 & u_{22} \end{bmatrix} \\ L & U \end{array}$$

$$\left\{\begin{array}{l} LUx = b \\ Ax = b \end{array}\right.$$

step1: Solve $Ly = b$

$$\begin{array}{c} L \\ \begin{bmatrix} 1 & 0 \\ m_{21} & 1 \end{bmatrix} \end{array} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = b$$

using Forward Substitution

$$A = LU$$

$$\boxed{\begin{array}{l} Ly = b \\ Ux = y \end{array}}$$

step2: use y from step 1 and solve

$$\begin{array}{c} U \\ \begin{bmatrix} 1 & 2 \\ 0 & u_{22} \end{bmatrix} \end{array} \begin{array}{c} x \\ \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{array} = \begin{array}{c} y \\ \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \end{array}$$

using Backward Substitution

Review inverse of a $\overset{2\times 2}{\text{matrix}}$ $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$