Math 4329

Thursday, April 23, 2020 4:34 PM

Last few lectures:

Developed methods $T_n(f) = h/2\, f(x_0) + h f(x_1) + \dots + f(x_n)\, h/2$

$S_n(f) = \dots$

Improve $E_n^S(f) = I - S_n(f)$ where $I = \int_a^b f(x)\,dx$

by ↑ number n of subintervals.

① $I = \int_a^b f(x)\,dx$ → Area/Quadrature

② $\tilde{I}(f)$ ← TILDE → Approxm to I formula

QUADRATURE FORMULA

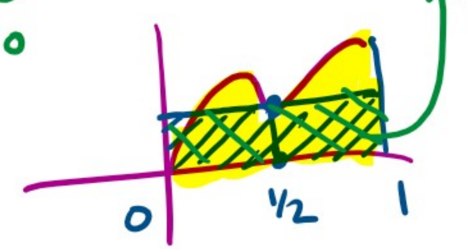
integration Rule/formula

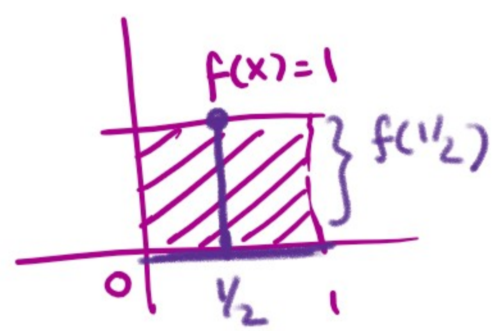
① Relax $x_0 = a$ & $x_n = b$

② $x_1, x_2, \dots x_{n-1}$ need not be equally spaced.

Example: Mid Point Rule

$$I = \int_0^1 f(x)\,dx \simeq (1-0) f(1/2)$$





midpt Rule: $\int_a^b f(x)\,dx \simeq (b-a)\, f\left(\frac{a+b}{2}\right)$

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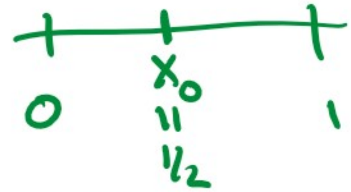
$a=0 \quad b=1$



Does $\tilde{I}(f) = f(1/2)$ give a good approximation to I ?

DoP: Degree of Precision of integratn rule $\tilde{I}$.

DoP of $\tilde{I}(f) = f(1/2)$ is said to be 1 if

$f(x) = 1 \quad \Rightarrow \quad \tilde{I}(f) = \int_0^1 f(x)\,dx$

$\Rightarrow$ Error $= 0$

and $f(x) = x^1 \quad \Rightarrow \quad \tilde{I}(f) = \int_0^1 x\,dx \Rightarrow$ Error $= I - \tilde{I}(f) = 0$

BUT $f(x) = x^2 \quad \Rightarrow$ Error $= I(f) - \tilde{I}(f) \neq 0$

Then, DoP $\tilde{I}(f)$ is 1

Verify the DoP of $\tilde{I}(f) = f(1/2)$ approximating

$$I(f) = \int_0^1 f(x)\,dx \quad \text{is } 1.$$

DoP = 1 means $f(x) = 1 \Rightarrow \tilde{I}(f) = I(f) \rightarrow$ error $= 0$

$f(x) = x \Rightarrow \tilde{I}(f) = I(f) \rightarrow$ error $= 0$

BUT $f(x) = x^2 \Rightarrow I(f) - \tilde{I}(f) \neq 0$

$I(f) \neq \tilde{I}(f)$

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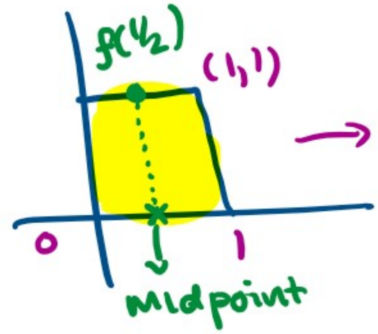
$f(x) = 1 \quad \int_0^1 1\,dx$



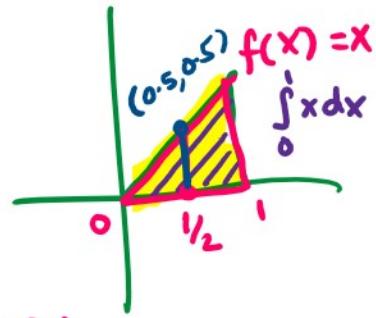
→ area of square = 1

$I(f) = \int_0^1 1\,dx = 1$

$\boxed{\tilde{I}(f) = f(1/2)}$ $\quad \tilde{I}(f) = f(1/2) = 1$

$I(f) = \tilde{I}(f) = 1$

$f(x) = x$ check $\boxed{\tilde{I}(f) \stackrel{?}{=} I(f)}$



$\tilde{I}(f) = f(1/2) = 1/2$

$I(f)$ = area under $f(x) = x$ bet. 0 & 1

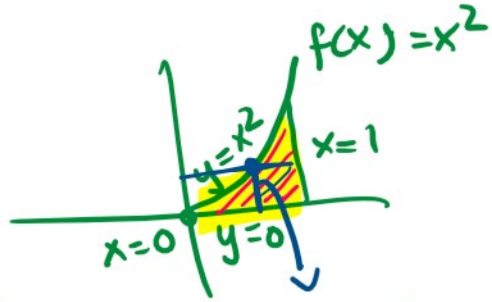
$= \frac{1}{2}$ Base * height $= \frac{1}{2} * 1 * 1 = 1/2$

$\hookrightarrow I(f) = \int_0^1 x\,dx = \frac{x^2}{2}\Big|_0^1 = 1/2$

$\underline{f(x) = x^2} \rightarrow \tilde{I}(f) = f(1/2) = 1/4$

$I(f) = \int_0^1 f(x)\,dx = \int_0^1 x^2\,dx = \frac{x^3}{3}\Big|_0^1$

$= 1/3$



$I(f) = 1/3$

formula for I: $\boxed{\int x^n\,dx = \frac{x^{n+1}}{n+1},\ n \geq 1}$

$\tilde{I}(f) = 0.25 \qquad I(f) = 1/3$

↳ underestimating area $1/3 = I(f)$

↳ underestimating area $I_3 = I(f)$

error $= 0.333 - 0.25 \neq 0$

$\Rightarrow$ $\hat{I}(f)$ has DoP 1 because

for

$$\left.\begin{array}{lll} f(x)=1 & \rightarrow & \hat{I}(f) = I(f) \\ f(x)=x^1 & \rightarrow & \hat{I}(f) = I(f) \end{array}\right\} \Rightarrow \text{DoP}=1$$

$$f(x)=x^2 \rightarrow \hat{I}(f) \neq I(f)$$

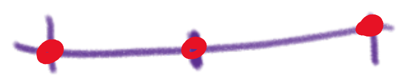
DoP $=3$ $\quad S_n(f) =$ Simpson's Rule has DoP $=3$ on any $[a,b]$.

means verify

$$\left.\begin{array}{lll} f(x)=1 & \Rightarrow & S_n(f) = I(f) \\ f(x)=x & \Rightarrow & S_n(f) = I(f) \\ f(x)=x^2 & \Rightarrow & S_n(f) = I(f) \\ f(x)=x^3 & \Rightarrow & S_n(f) = I(f) \end{array}\right\} \text{DoP}=3$$

BUT $\quad f(x)=x^4 \Rightarrow S_n(f) \neq I(f)$

Fix $a=3$ $\quad b=7$ $\quad n=2$



$n=2$ $\quad h = \dfrac{b-a}{n} = \dfrac{7-3}{2} = 2$ $\qquad x_0 = 3 \quad \underset{x_1}{5} \quad 7 = x_2$

$$S_2(f) = \frac{h}{3} f(3) + \frac{4}{3}h\, f(5) + h/3\, f(7)$$

$$I(f) = \int_3^7 f(x)\,dx$$

useful: $\displaystyle\int_a^b x^n\,dx = \left.\frac{x^{n+1}}{n+1}\right|_{x=a}^{b}$

$$= \frac{b^{n+1}}{n+1} - \frac{a^{n+1}}{n+1}$$

$f(x)=1 \rightarrow \quad h=2$

$S_2(f) = \frac{2}{3} f(3) + \frac{4}{3}(2) f(5) + \frac{2}{3} f(7)$

$S_2(f) = 2/3 + 8/3 + 2/3 = 12/3 = 4$

$I(f) = \int_3^7 1 \, dx = x \Big|_{x=3}^{7} = 7-3 = 4$

$I(f) - S_2(f) = 0$

$f(x)=x \rightarrow S_2(f) = \frac{2}{3} f(3) + \frac{8}{3} f(5) + \frac{2}{3} f(7)$

$= \frac{2}{3} * 3 + \frac{8}{3} * 5 + \frac{2}{3} * 7$

$= \frac{6+40+14}{3} = \frac{60}{3} = 20$

$f(x)=x \rightarrow I(f) = \int_3^7 x \, dx = \frac{x^2}{2} \Big|_{x=3}^{7} = \frac{49-9}{2} = 20$

$f(x)=x \quad I(f) - S_2(f) = 0$

$f(x)=x^2 \quad S_2(f) = 2/3 \; 3^2 + \frac{8}{3} 5^2 + \frac{2}{3} 7^2$

$= \frac{18}{3} + \frac{200}{3} + \frac{98}{3} = \frac{316}{3}$

$I(f) = \int_3^7 x^2 dx = \frac{1}{3} x^3 \Big|_{x=3}^{7} = \frac{1}{3}\left(7^3 - 3^3\right) = \frac{316}{3}$

$f(x)=x^2 \quad I(f) - S_2(f) = 316/3 - 316/3 = 0$

$f(x)=x^3 \quad S_2(f) = 2/3 f(3) + 8/3 f(5) + 2/3 f(7)$

$= 2/3 (3)^3 + \frac{8}{3} * 5^3 + \frac{2}{3} * 7^3$

$= \quad 1000 + 686 \quad 1740 = 580$

$$= \frac{54 + 1000 + 686}{3} = \frac{1740}{3} = 580$$

$$I(f) = \int_3^7 x^3\,dx = \frac{1}{4}(7^4 - 3^4) = 580$$

DoP is atleast 3.

$f(x) = x^4$ $\quad S_2(f) = 2/3 * 3^4 + \frac{8}{3} * 5^4 + \frac{2}{3} * 7^4$

$$= \frac{162 + 5000 + 4802}{3} = \frac{9964}{3} = 3321.3$$

$$I(f) = \int_3^7 x^4\,dx = \frac{16564}{5} = 3312.8$$

$S_2(f) = 2/3\, f(3) + 8/3\, f(5) + 2/3\, f(7)$

DoP = 3 because $f(x) = 1 \Rightarrow I(f) - S_2(f) = 0$

$f(x) = x, x^2, \underline{\underline{x^3}} \Rightarrow I(f) - S_2(f) = 0$

BUT $f(x) = x^4 \Rightarrow I(f) - S_2(f) \neq 0$

$$\int_0^4 f(x)\,dx \approx 2f(1) + 2f(3) = \hat{I}(f)$$

↳ approximates

$\hat{I}(f) = 2f(1) + 2f(3)$ has DoP 1 on [0,4]

$\boxed{f(x)=1} \Rightarrow \hat{I}(f) = 4,\ I(f) = \int_0^4 1\,dx = 4 \Rightarrow \hat{I}(f) = I(f)$

$\boxed{f(x) = x} \Rightarrow \hat{I}(f) = 2*1 + 2*3$

$= 8$

$$I(f) = \int_0^4 x\,dx = \frac{x^2}{2}\Big|_0^4 = 16 = 8$$

$= 8$

$= \frac{16}{2} = 8$

$$\boxed{I(f) - \hat{I}(f) = 8 - 8 = 0}$$

$$\boxed{f(x) = x^2} \Rightarrow \hat{I}(f) = 2 * 1^2 + 2 * 3^2$$

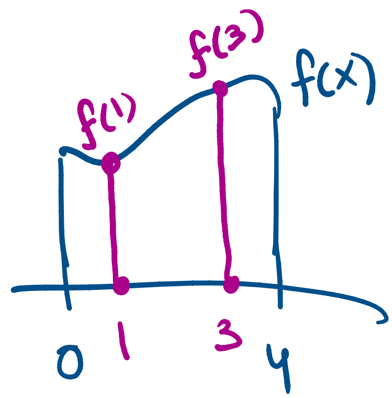
$$= 2 + 18 = 20$$

$$I(f) = \int_0^4 x^2\,dx = \frac{4^3}{3} = \frac{64}{3}$$

$$\boxed{I(f) \neq \hat{I}(f)}$$

$$\boxed{\text{DoP} = 1}$$

$$\hat{I}(f) = 2f(1) + 2f(3)$$



1, 3 → nodes
2, 2 → weights of formula.