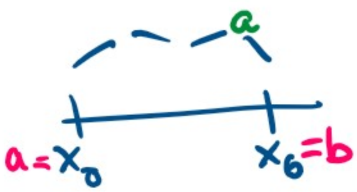
Math4329 - class on April 21st

Thursday, April 16, 2020 6:17 PM

Simpson's Rule

$$I = \int_a^b f(x)\,dx = \int_{x_0}^{x_2} f(x)\,dx + \int_{x_2}^{x_4} f(x)\,dx + \int_{x_4}^{x_6} f(x)\,dx$$

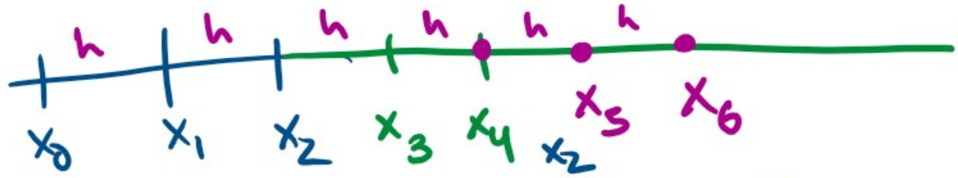


$$\simeq \int_{x_0}^{x_2} \underbrace{p_2^{(0)}(x)}_{\text{deg 2 poly}}dx + \int_{x_2}^{x_4} \underbrace{p^{(2)}(x)}_{\text{deg 2 poly}}dx + \int_{x_4}^{x_6} \underbrace{p^{(4)}(x)}_{\text{deg 2}}dx$$

Quad. poly interp. x_0, x_1, x_2

Quad poly interpolating x_2, x_3, x_4

Quad poly interpolating x_4, x_5, x_6.



$$\int_{x_0}^{x_2} f(x_0)\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}dx + \int_{x_0}^{x_2} f(x_1)\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}dx$$

$$+ \int_{x_0}^{x_2} f(x_2)\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}dx$$

($x_0 \leftarrow 0$)

Lagrange Quad poly for x_0, x_1, x_2

By making a "clever" substitution* we can solve each of the integrals painlessly!

How? $\int^{x_2} \dfrac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} dx$ * Let $u = x - x_0$

then $x = x_0 \Rightarrow u = 0$

How? $\int_{x_0} \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}\,dx$ * Let $u = \ldots$

then $x = x_0 \Rightarrow u = 0$

$x = x_2 \Rightarrow u = 2h$

Also $x - x_1 = u + x_0 - x_1 = u - h$

$x - x_2 = u + x_0 - x_2 = u - 2h$

$$\int_0^{2h} \frac{(u-h)(u-2h)}{2h^2}\,du = \int_0^{2h} \frac{u^2 - 3hu + 2h^2}{2h^2}\,du$$

$$= \frac{1}{2h^2}\left(\frac{u^3}{3} - 3h\frac{u^2}{2} + 2h^2 u\right)\Bigg|_{u=0}^{2h}$$

$$= \frac{1}{2h}\left(\frac{8h^3}{3} - \frac{3}{2}h\overset{12h^3}{(4h^2)} + 4h^3\right)$$

$$= \frac{1}{2h^2}\left(\frac{8h^3}{3} - \frac{12h^3}{2} + \frac{24h^3}{6}\right)$$

$$= \left(\frac{16 - 36 + 24}{12h^2}\right)h^3 = 3h$$

Similarly using the same substitution, $u = x - x_0$,

$$f(x_1)\int_{x_0}^{x_2} \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}\,dx = -\frac{f(x_1)}{h^2}\int_0^{2h} u(u-2h)\,du = \frac{4f(x_1)h}{3} \text{ and}$$

$$f(x_2)\int_{x_0}^{x_2} \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}\,dx = h/3$$

$$f(x_2)\int_{x_0}^{x_2} \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}\,dx = h/3$$

Thus
$$\boxed{\int_{x_0}^{x_2} p^{(0)}(x)\,dx = \frac{h}{3}f(x_0) + \frac{4}{3}h\,f(x_1) + \frac{h}{3}f(x_2)}$$

interpolating $(x_0, f(x_0))\ (x_1, f(x_1))\ (x_2, f(x_2))$

$$\int_{x_2}^{x_4} p^{(0)}(x)\,dx = \frac{h}{3}f(x_2) + \frac{4h}{3}f(x_3) + \frac{h}{3}f(x_4)$$

$$\int_{x_4}^{x_6} p^{(0)}(x)\,dx = \frac{h}{3}f(x_4) + \frac{4h}{3}f(x_5) + \frac{h}{3}f(x_6)$$

$$\int_{a=x_0}^{b=x_6} f(x)\,dx \approx \int_{x_0}^{x_2} p^{(0)}(x)\,dx + \int_{x_2}^{x_4} p^{(0)}(x)\,dx + \int_{x_4}^{x_6} p^{(0)}(x)\,dx$$

$$h/3\, f(x_0) + \frac{4h}{3}f(x_1) + h/3\, \underline{\underline{f(x_2)}} +$$

$$h/3\, \underline{\underline{f(x_2)}} + \frac{4h}{3}f(x_3) + {*}\,h/3\, \underline{\underline{f(x_4)}} +$$

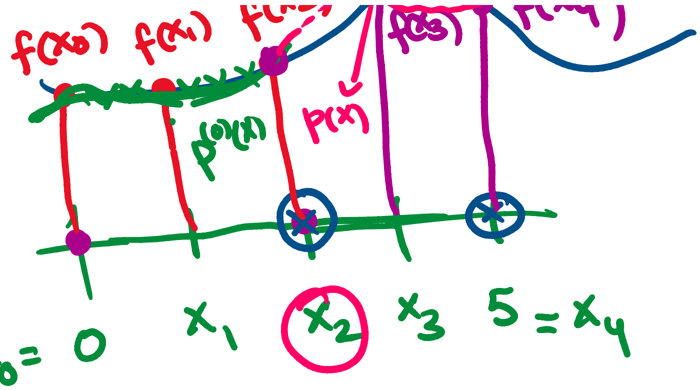
$$*\ h/3\, \underline{\underline{f(x_4)}} + \frac{4h}{3}f(x_5) + h/3\, f(x_6)$$

$$S_6(f) = h/3\, f(x_0) + 4h/3\, f(x_1) + 2\frac{h}{3}f(x_2) +$$

$$\frac{4h}{3}f(x_3) + 2h/3\, f(x_4) + \frac{4h}{3}f(x_5)$$

$$+\ h/3\, f(x_6)$$

$+ \ h/3 \ f(x_6)$



$S_4(f)$

$$\int_{x_0}^{x_2} p^{(0)}(x)\,dx = \frac{h}{3} f(x_0) + \frac{4h}{3} f(x_1) + \frac{h}{3} f(x_2)$$

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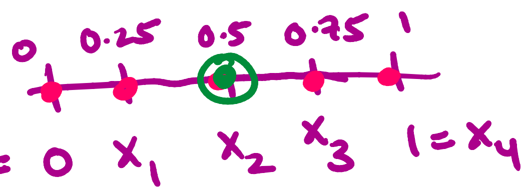
4(c) $\quad f(x) = \sqrt{x}\, e^x \quad 0 \le x \le 1$

Determine

$\hookrightarrow S_4(f)$



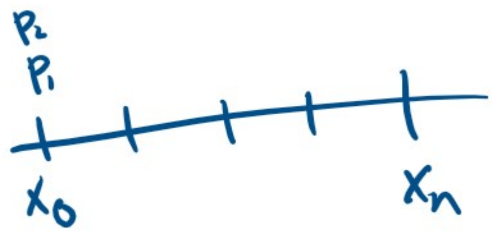
fomula: $h = \frac{b-a}{n} = 1/4 = 0.25 \quad f(x) = \sqrt{x}\, e^x$

$S_4(f) = h/3\ f(0) + 4h/3\ f(0.25) + 2h/3\ f(0.5) + 4h/3\ f(0.75) + h/3\ f(1)$

$$S_4(f) = \frac{0.25}{3}\sqrt{0}\, e^0 + \frac{4}{3} * 0.25 \sqrt{0.25}\, e^{0.25} + \frac{2*0.25}{3}\sqrt{0.5}\, e^{0.5} + \frac{4*0.25}{3}\sqrt{0.75}\, e^{0.75} + \frac{0.25}{3}\sqrt{1}\, e^1$$

$= 0 + 0.214 \ldots$

$= 1.246$



#4: $I = \int_0^1 \sqrt{1+f'(x)^2}\, dx$ arc length of $f(x)$ bet 0 & 1

approxm. arc length I

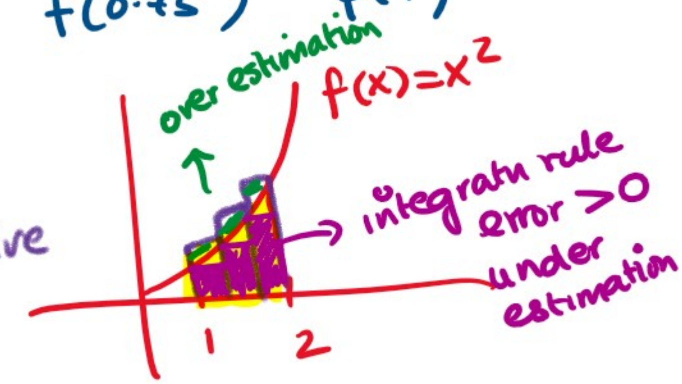
using $T_4(f)$, $S_4(f)$.

$$S_4(f) = \frac{h}{3}\sqrt{1+f'(0)^2} + \frac{4h}{3}\sqrt{1+f'(.25)^2} + \frac{2h}{3}\sqrt{1+f'(0.5)^2} + \frac{4h}{3}\sqrt{1+f'(0.75)^2} + \frac{h}{3}\sqrt{1+f'(1)^2}$$

$f(0)$ $f(0.25)$ $f(0.5)$ $f(0.75)$ $f(1)$

#3 $I - T_4(f) < 0$

if error = True Value − Approx. value is negative



Out of scope but imp:

without calculating find out the number of subintervals n needed so that

Subintervals n needed so that

$$|I - T_n(f)| < 10^{-6}$$

$$|I - S_n(f)| < 10^{-6}.$$

$$E_n^T(f) = -\frac{h^2(b-a)}{12} f''(c_n)$$

c_n is unknown between a & b

$$h = (b-a)/n = 0.1$$

$$E_n^S(f) = -\frac{h^4(b-a)}{180} f^{(IV)}(c_n)$$

$E_n^T(f) = (0.1)^2 \to 0.01$ $\qquad E_n^S(f) \approx 0.0001$

Goal: Develop other methods for computing

$$I = \int_a^b f(x)\,dx$$

$$\approx I_n(f) := w_1 f(x_1) + w_2 f(x_2) + \cdots w_k f(x_k)$$

weights

NODES

Degree of Precision (DoP)