Math-4329

Tuesday, April 14, 2020 4:23 PM

Num. Derivatives

population Growth $y(t) \to$ pop^n at time t

$$\left\{ \begin{aligned} \frac{dy}{dt} &= ry \\ y(0) &= 3 \end{aligned} \right\}$$ solve $\frac{dy}{dt} = ry$

$$D_h^+ y(t) = ry(t)$$

$$\frac{y(t+h) - y(t)}{h} = ry(t)$$

$t = 0 \quad h = 2$

$$\frac{y(0+2) - y(0)}{0+2-0} = ry(0)$$

Days time	$y(t)$
0 = 17th March	$3 = y(0)$
2 = 19th March	$6 = y(2)$
5 = 24th March	10

Last lecture introduced forward diff operator

$$D_h^+ f(x) = \frac{f(x+h) - f(x)}{x+h-x} = \frac{\text{change in } y}{\text{change in } x}$$

Backward diff operator

$$D_h^- f(x) = \frac{f(x) - f(x-h)}{x-(x-h)} = \frac{f(x) - f(x-h)}{h}$$

$D_h^+ f(x)$ & $f'(x)$

$$f'(x) = D_h^+ f(x) + O(h)$$

$O(h) \to$ terms which are multiplied by h

$$h \to 0 \qquad D_h^+ f(x) \longrightarrow f'(x)$$

$$f'(x) = D_h^- f(x) + O(h)$$

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x	$f(x)$	$D_h^+ f(x)$	$D_h^- f(x)$
$x=-1$	-0.45	0.45	NA
$x=0$	0	0.5	0.45
$x=1$	0.5	NA	0.5

$h=1$

$h=1$

$$D_h^- f(-1) = \frac{f(-1)-f(-1-1)}{-1-(-1-1)} = \frac{f(-1)-f(-2)}{1}$$

No info available at $x=-2$

$$D_h^+ f(-1) = \frac{f(-1+h)-f(-1)}{-1+h-(-1)}$$

$h=1$

$$D_1^+ f(-1) = (f(0)-f(-1))/h = \frac{0-(-0.45)}{1} = 0.45$$

Know the value of $f'(x)$ how do $D_h^+ f(x)$, $D_h^- f(x)$ behave?

x	$f(x)=\cos\pi x$	$f'(x)=-\pi\sin\pi x$	$D_h^+ f(x)$	$D_h^- f(x)$
0.5	0	$-\pi$		
1	-1	0		
2	1	0		

$x=0.5$ $\quad f(0.5)=0 \quad f(x)=\cos\pi x$

$h=0.5$

$$D_h^+ f(0.5) = \frac{f(0.5+h)-f(0.5)}{h} = \frac{\overset{\cos\pi}{f(1)} - \overset{\cos\pi/2}{f(0.5)}}{0.5}$$

$$D_h^+ f(0.5) = \frac{-1-0}{0.5} = -2$$

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$$f(x) = \cos \pi x \qquad f'(x) = -\pi \sin \pi x$$

$$f'(0.5) = -\pi \sin(\pi/2) = -\pi$$

$D_h^- f(0.5)$ = Not applicable.

$$h=0.5 \quad D_h^- f(1) = \frac{\overset{\cos\pi}{f(1)} - \overset{\cos\pi/2}{f(1-h)}}{1-(1-h)} = \frac{-1-0}{h} = -\frac{1}{0.5} = -2$$

$$f'(1) = -\pi \sin \pi = 0 \qquad D_h^- f(1) = -2$$

$$f(x) = \cos \pi x$$

$$h=1.5 \quad D_h^+ f(0.5) = \frac{f(0.5+h) - f(0.5)}{0.5+h-0.5}$$

$$= \frac{f(0.5+1.5) - f(0.5)}{h}$$

$$= \frac{\overset{\cos(2\pi)}{f(2)} - \overset{\cos(\pi/2)}{f(0.5)}}{1.5}$$

$$D_h^+ f(0.5) = 1/1.5 = 2/3 \qquad f'(0.5) = -\pi$$

x	$f(x) = \cos \pi x$

you can pick h randomly

$$D_h^+ f(0.25) = \frac{f(0.25+h) - f(0.25)}{0.25+h-0.25}$$

$$D_h^+ f(0.25) = \frac{\cos(\pi(0.25+h)) - \cos(0.25\pi)}{}$$

$$D_h^+ f(0.25) = \frac{\cos(\pi(0.25+h)) - \ldots}{h}$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h\to 0} D_h^+ f(x)$$

using polynomial $p_2(x)$ interpolates

x_i	y_i
$x_0 = x^* - h$	$y_0 = f(x^* - h)$
$x_1 = x^*$	$y_1 = f(x^*)$
$x_2 = x^* + h$	$y_2 = f(x^* + h)$

$$f(x) \simeq p_2(x)$$

$x_1 = x^*$ point where we want to approximate $f'(x^*)$

$$f'(x^*) \simeq p_2'(x^*)$$

$$= D_h^c f(x^*)$$ $\hookrightarrow$ Central Difference Operator

$$= \frac{f(x^*+h) - f(x^*-h)}{x^*+h - (x^*-h)}$$

$$D_h f(x^*) = D_h^c f(x^*) = \frac{f(x^*+h) - f(x^*-h)}{2h}$$

x 1 $f(x) = -x^2$ | $h=1$ $\boxed{x^* = 1}$ $D^c f(x^*) = f(1+h)^{2} - f(1-h)^{0}$

x	$f(x)=-x^2$
0	0
1	-1
2	-4

$$D_h^c f(x^*) = \frac{f(1+h) - f(1-h)}{1+h-(1-h)}$$

$$D_h^c f(1) = \frac{-4-0}{2} = -2 = f'(1)$$

$f(x) = -x^2$

$f'(x) = -2x$

$f'(1) = -2$

$$D_h^c f(x^*) = \frac{f(x^*+h) - f(x^*-h)}{2h} \approx f'(x^*)$$

$f(x) \approx p_2(x)$ interpolates $\underset{x_0}{x^*-h}, \underset{x_1}{x^*}, \underset{x_2}{x^*+h}$

$f'(x) = -2x \leftarrow f(x) = -x^2$ (Quadratic degree 2 poly)

$x^* = 2 \quad h = 1 \quad f(x) = -x^2 \qquad 2+1=3 \quad 2-1=1$

$$D_h f(2) = D_h^c f(2) = \frac{\overset{-3^2}{f(2+h)} - \overset{(-1^2)}{f(2-h)}}{2+h-(2-h)} = \frac{-9-(-1)}{2} = -\frac{8}{2} = -4$$

$f'(2) = -2(2) = -4$

$f'(x) = D_h^+ f(x) + O(h) \Rightarrow$ Linear convergence of $D_h^+ f(x)$ to $f'(x)$

$f'(x) = D_h^- f(x) + O(h) \Rightarrow$ $D_h^- f(x)$ to $f'(x)$

$f'(x) = D_h f(x) + O(h^2)$ Quadratic Convergence of $D_h f(x)$ to $f'(x)$

$D_h f(x) = D^c f(x)$

$f'(x) \to$ one prefers $D_h f(x) \equiv D_h^c f(x)$

What about $f''(x) = ?$ using discrete set of data (x_0, y_0) (x_1, y_1) (x_2, y_2)

a) Direct formula involving $f(x+h), f(x), f(x-h)$

$$D_h^2 f(x) = \frac{f(\underset{x_2}{x+h}) - 2f(\underset{x_1}{x}) + f(\underset{x_0}{x-h})}{h^2}$$

b) Based on $\underline{D_h^+ f(x)}$ & $\underline{D_h^- f(x)}$ $D_h^2 f(x) = \frac{D_h^+ f(x) - D_h^- f(x)}{h}$

$$D_h^2 f(x) = \frac{D_h^+ f(x) - D_h^- f(x)}{h}$$

$$= \frac{\frac{1}{h}\left(f(x+h) - f(x)\right) - \left(\frac{f(x) - f(x-h)}{h}\right)}{h}$$

$$= \frac{f(x+h) - f(x) - f(x) + f(x-h)}{h^2}$$

$=$ DIRECT FORMULA

Order of Convergence of $D_h^2 f(x) = ?$

Discuss this in next lecture (Extra credit)

hint: Taylor Expansion

$$f(x+h) \simeq f(x) + hf'(x) + \frac{h^2 f''(x)}{2} + \frac{h^3 f'''(x)}{6} + h^4 f''''(x)/4!$$

$$+ \quad f(x-h) \simeq f(x) - hf'(x) + \frac{h^2 f''(x)}{} - \frac{h^3 f'''(x)}{}$$

$$+ \quad f(x-h) \approx f(x) - hf'(x) + \frac{h^2 f''(x)}{2} - \frac{h^3 f'''(x)}{6}$$

$$+ h^4 f''''(x)/4!$$

$$f(x+h) + f(x-h) \approx 2f(x) + \frac{h^2}{2} f''(x) + \frac{h^2}{2} f''(x) +$$

$$\frac{h^4}{4!} f''''(x) + \frac{h^4}{4!} f''''(x)$$

$$f(x+h) + f(x-h) - 2f(x) \approx 2f(x) - 2f(x) + 2\frac{h^2}{2!} f''(x) + 2\frac{h^4}{4!} f''''(x)$$

$$\frac{f(x+h) - 2f(x) + f(x-h)}{h^2} \approx \frac{\cancel{h^2}\left(f''(x) + \frac{2h^2}{4!} f''''(x) \right)}{\cancel{h^2}}$$

$$D_h^2 f(x) \approx f''(x) + \frac{2h^2}{4!} f''''(x)$$

$$f''(x) \approx D_h^2 f(x) - \underbrace{\frac{2h^2}{4} f''''(x)}_{O(h^2)}$$

$$D_h^2 f(x) \longrightarrow f''(x)$$ as $h \to 0$ at Quadratic speed