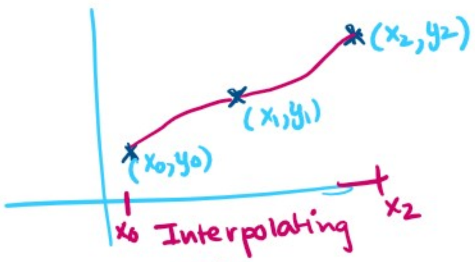
Math 4329

Tuesday, April 7, 2020 3:09 PM

Spline



Interpolating poly

Lagrange & Newton D.D poly $\qquad p_2(x_i) = y_i$

Spline $\rightarrow$ achieve more that Lagrange or Newton D.D poly.

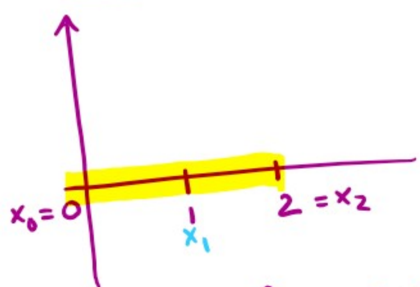
$\rightarrow$ smooth

$\boxed{S(x) \text{ function continuous}}$

$S'(x)$ & $S''(x)$ are continuous between x_0, x_1, x_2

$\boxed{S(x_i) = y_i}$

$$S(x) = \begin{cases} S_1(x) & 0 \le x \le 1, \\ S_2(x) & 1 < x \le 2. \end{cases}$$



$$S(x) = \begin{cases} S_1(x) = (x-1)^3 & 0 \le x \le 1 \\ S_2(x) = 2(x-1)^3 & 1 < x \le 2. \end{cases}$$

Check if $S(x)$ is a Spline.

Note: y_i values need not be prescribed when checking if $S(x)$ is a spline or not.

Condns for spline:

Condn 1: $S(x)$ is continuous on $[0, 2]$

that means $S_1(x)$ & $S_2(x)$ at $x = 1$ match up.

$S(x) = (x-1)^3 \quad 0 \le x \le 1 \rightarrow \lim\limits_{x \to 1^-} S(x) = \lim\limits_{x \to 1^+} S(x)$

$\lim\limits_{x \to 1^-} S(x) = \lim\limits_{x \to 1^-} S_1(x) = (1-1)^3 = 0$

$\lim\limits_{x \to 1^+} S(x) = \lim\limits_{x \to 1^+} \underbrace{2(x-1)^3}_{S_2(x)} = 2(1-1)^3 = 0$

$= \lim S(x).$

$\lim_{x \to 1^+}$ $\lim_{x \to 1^-} \underbrace{}_{s_2(x)}$

Condn 1 is satisfied since $\lim_{x \to 1^-} s(x) = \lim_{x \to 1^+} s(x)$.

Condn 2: $s'(x)$ is cts on $[0,2]$

$$s'(x) = \begin{cases} s_1'(x) & 0 \le x \le 1, \\ s_2'(x) & 1 < x \le 2. \end{cases}$$

$s_1(x) = (x-1)^3$, $s_2(x) = 2(x-1)^3$

$$s'(x) = \begin{cases} 3(x-1)^2 & 0 \le x \le 1 \\ 6(x-1)^2 & 1 < x \le 2 \end{cases}$$ gluing point

$s_1'(x)$ is cts for $0 < x < 1$

$s_2'(x)$ is cts for $1 < x < 2$

check at $x = 1$?

$$\lim_{x \to 1^-} s'(x) \stackrel{?}{=} \lim_{x \to 1^+} s'(x)$$

$$\lim_{x \to 1} 3(x-1)^2 \quad (s_1'(x)) \qquad \lim_{x \to 1} s_2'(x)$$

$$\lim_{x \to 1} 6(x-1)^2$$

$3*0$ $\qquad$ $\|$

0 $\qquad$ 0

True!

Condn 3: $s''(x)$ is continuous on $[0,2]$

$$s''(x) = \begin{cases} s_1''(x) = 6(x-1) & 0 \le x \le 1 \\ s_2''(x) = 12(x-1) & 1 < x \le 2 \end{cases}$$

$s_1''(x)$ & $s_2''(x)$ are continuous for $0 < x < 1$ & $1 < x < 2$ respectively.

Now check for cty of $s''(x)$ at $x = 1$

$$\left.\begin{aligned} s_1''(1) &= 6*0 = 0 \\ s_2''(1) &= 12*0 = 0 \end{aligned}\right\} \quad \lim_{x \to 1^-} s''(x) = 0 = \lim_{x \to 1^+} s''(x)$$

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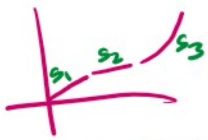
$$s(x) = \begin{cases} x^3 \quad s_1(x) & 0 \le x \le \boxed{1}\,? \\ 2x-1 \quad s_2(x) & \boxed{1} \stackrel{?}{<} x < 2\ ?? \end{cases}$$

$$S(x) = \begin{cases} x^3 \quad s_1(x) & 0 \le x \le \boxed{1} \\ 2x-1 \quad s_2(x) & \overset{?}{\boxed{1}} < x < 2 \ ?? \\ 3x^2-9 \quad s_3(x) & \underset{??}{2} \le x \le 3 \end{cases}$$



cty of $S(x)$

$s_1(x) = x^3$	$x=1$	$s_2(x) = 2x-1$	$x=2$	$s_3(x) = 3x^2-9$
$s_1(1) = 1$		$s_2(1) = 2-1 = 1$		$s_3(2) = 12-9 = 3$
$s_1(1) =$		$s_2(1)$ ✓		
		$\boxed{s_2(2) = 4-1 = 3}$		$\boxed{= s_3(2)}$

$\Rightarrow$ $S(x)$ is cts on $[0,3]$ condn 1 holds

cty of $S'(x)$

$s_1'(x) = 3x^2$	$\overset{?}{\boxed{x=1}}$	$s_2'(x) = 2$	$\overset{?}{\boxed{x=2}}$	$s_3'(x) = 6x$
$s_1'(1) = 3$		$s_2'(1) = 2$		$s_3'(1) = 6$
$s_1'(1)$	$\neq$	$s_2'(1)$		
		$s_2'(1)$	$\neq$	$s_3'(1)$

$\Downarrow$

NOT A SPLINE!

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$$S(x) = \begin{cases} s_1(x) := x^3 + 2x^2 + 1 & 1 \le x \le 2 \\ s_2(x) := -2x^3 + \beta x^2 - 36x + 25 & 2 < x \le 3 \end{cases}$$

For what value of β is $S(x)$ a spline?

find β so that:

$S(x)$, $S'(x)$ & $S''(x)$ are cts on $[1,3]$.

choose β so that

$$\lim_{x\to 2^-} S(x) = \lim_{x \to 2^+} S(x)$$

$$\lim_{x\to 2^-} S'(x) = \lim_{x \to 2^+} S'(x)$$

& $$\lim_{x\to 2^-} S''(x) = \lim_{x \to 2^+} S''(x$$

Helpful Tip: start with $S''(x)$.

$$S(x) = \begin{cases} s_1(x) = x^3 + 2x^2 + 1 & 1 \le x \le 2 \\ s_2(x) = -2x^3 + \beta x^2 - 36x + 25 & 2 < x \le 3. \end{cases}$$

$$S(x) = \begin{cases} S_1(x) = x^3 + 2x^2 + 1 & 1 \le x \le 2 \\ S_2(x) = -2x^3 + \beta x^2 - 36x + 25 & 2 < x \le 3. \end{cases}$$

Mistake by me Please disregard the solution below:

cty of $S(x)$ $\lim_{x \to 2^-} S(x) = S_1(1) = S_2(1) = \lim_{x \to 2^+} S(x)$

$S_1(1)$: $1^3 + 2 + 1 = 4$

$S_2(1)$: $\boxed{-2 + \beta - 36 + 25}$

choose β so that $S_2(1) = S_1(1)$

$$\underbrace{-2}_{-2} + \beta \underbrace{- 36 + 25}_{-11} = 4$$

$$\beta = 4 + 13 = 17$$

$\boxed{\beta = 17}$ → need to verify that $S_1'(2) = S_2'(2)$ for $\beta = 17$

$S_1''(2) = S_2''(2)$

condn of cty of $S'(x)$:

$S_2(x) = -2x^3 + \beta x^2 - 36x + 25$, $\beta = 17$

$S_2'(x) = -6x^2 + 2\beta x - 36$, $\beta = 17$

$S_2'(2) = -24 + 4\beta - 36$

$= -60 + 4*17 = 8$

$S_1'(2) = ?$

$S_1(x) = x^3 + 2x^2 + 1$

$S_1'(x) = 3x^2 + 4x$

$S_1'(2) = 3*4 + 8 = 12 + 8 = 20$

$S_1'(2) = 20 \neq S_2'(2) = 8$

<u>Notice</u> $\beta = 17$ guarantees cty of $S(x)$ but not of $S'(x)$ or $S''(x)$.

We should make sure that we find β so that $S''(x)$ is continuous on $[1, 3]$

$\Rightarrow$ $S'(x)$, $S(x)$ are also cts on $[1, 3]$.

$S_1(x) = x^3 + 2x^2 + 1$

$S_1'(x) = 3x^2 + 4x$

$S_1''(x) = 6x + 4$

$S_2(x) = -2x^3 + \beta x^2 - 36x + 25$

$S_2'(x) = -6x^2 + 2\beta x - 36$

$S_2''(x) = -12x + 2\beta$

To guarantee cty of $S''(x)$ on $[1, 3]$,

we need $S_1''(2) = S_2''(2)$

To guarantee cty of $s''(x)$ on $[1,3]$,
we need $s_1''(2) = s_2''(2)$

$6*2+4$ $-12*2+2\beta$

$16 = -24+2\beta$

find β so that

$$-24+2\beta = 16$$
$$\beta = \tfrac{1}{2}(16+24)$$
$$\beta = 20$$

check: $\beta=20$ $-24+40=16$

Verify $s'(x)$ is cts

p

Another example: find a, b, c & d so that $s(x)$ is a spline.

$$s(x) = \begin{cases} s_1(x) = (x+1)^3 & -2 \le x \le -1 \\ s_2(x) = ax^3+bx^2+cx+d & -1 < x < 1 \\ s_3(x) = (x-1)^2 & 1 \le x \le 2. \end{cases}$$

Gluing Points : $-1, 1$

Continuity of s_1, s_2, s_2 & s_3

use cty conditions to obtain a relation bet. a, b, c & d

cty of $s(x)$ at $x=-1$.

$$\lim_{x\to -1^+} s(x) = \lim_{x\to -1^-} s(x) = \underbrace{(-1+1)^3}_{s_1(-1)} = 0$$

$s_2(-1)$

$$\boxed{-a+b-c+d = 0} \to 1$$

cty of $s(x)$ at $x=1$

$$\lim_{x\to 1^-} s(x) = \lim_{x\to 1^+} s(x) = s_3(1) = (1-1)^2 = 0$$

$s_2(1)$

$\lim_{x \to 1^-} \qquad \lim_{x \to 1^+}$

$$\boxed{\overset{s_2(1)}{a+b+c+d} = 0} \to (2)$$

use cty of $s'(x)$:

$$\lim_{x \to -1^+} s'(x) = \lim_{x \to -1^+} s'(x) = \lim_{x \to -1^+} s_1'(x) \qquad 3(x+1)^2$$

$$= 3(-1+1)^2 = 0$$

$s_2(x) = ax^3+bx^2+cx+d$

$s_2'(x) = 3ax^2+2bx+c$

$s_2'(-1) = 3a-2b+c = s_1'(-1) = 0$

$$\boxed{3a-2b+c = 0} \to (3)$$

$$\lim_{x \to 1^-} s'(x) = \lim_{x \to 1^+} s'(x)$$

$$s_2'(1) = s_3'(1)$$

$s_3(x) = (x-1)^2$

$s_3'(x) = 2(x-1)$

$s_3'(1) = 0$

$x=1$

$$\boxed{3a+2b+c = 0} \to (4)$$

Note: $\underbrace{s_1''(-1) = s_2''(-1)}$ and $\underbrace{s_2''(1) = s_3''(1)}$

$s_2(x) = ax^3+bx^2+cx+d$

$s_2'(x) = 3ax^2+2bx+c$

$s_2''(x) = 6ax+2b \to s_2''(-1) = s_1''(-1) = 0$

$$\boxed{-6a+2b = 0} \to (5)$$

using $s_2''(1) = s_3''(1)$

$$\boxed{6a+2b = 2} \to (6)$$

$\boxed{6a + 2b \quad = 2} \rightarrow$ 6

6 equations & 4 unknowns how to solve?

solve the equations with least number of unknowns first that is $\left\{\begin{array}{l} 6 \ \& \ 5. \\ 4 \ \& \ 3. \\ 2 \ \& \ 1 \end{array}\right\}$ → should suffice!

Next class → we solve them!