Math 4329

Thursday, March 5, 2020 4:30 PM

Design of Iterative Methods → ONE-STEP FIXED POINT ITERATIVE Method

find α: $f(\alpha)=0$ ⟶ Guess x_0, construct x_1 based on a formula.

$x_2, x_3 \ldots \longrightarrow \alpha$

$|x_{n+1}-x_n| < \varepsilon$

e.g.: Guess x_0,

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} \ldots$$

Root finding for $f(x)=0$ ⟶ fixed pt of $g(x)$

$$f(\alpha)=0 \longrightarrow \underbrace{cf(\alpha)+\alpha}_{g(\alpha)} = \alpha, \; c \neq 0$$

$$x - 1(x^2-5) = 0 + x$$

I1: $\underbrace{x_n + (5-x_n^2)}_{g_1(x_n)} = x_{n+1}$

$$x_{n+1} = g_1(x_n)$$

How to design our own algo?

$\underbrace{x^3-1}_{f(x)}=0$ multiply by $c \neq 0$ and add x to both sides

$$x + c(x^3-1) = x + 0$$

$f(x)$

$x+c(x^3-1)$

$$\boxed{x_n + c(x_n^3 - 1) = x_{n+1}}$$

$$x_{n+1} = \underbrace{x_n + c(x_n^3-1)}_{g(x_n)}$$

Ques: Does: $x_{n+1} = g(x_n)$ $\underbrace{\text{converge to } \alpha = 1}_{\text{assume } x_0 \text{ is close enough to } \alpha.}$?

Theory tells: if $-1 < g'(\alpha) < 1$

$c \to$ unknown nonzero number "CONTROL PARAMETER"

$$g(x) = x + c(x^3-1)$$

$$g'(x) = 1 + c(3x^2)$$

$x \to \alpha$ $\quad \alpha = 1$

$$g'(\alpha) = 1 + 3c$$

In order to have $-1 < g'(\alpha) < 1$

we need $-1 < 1+3c < 1$ we don't know c.

we want to prescribe a value of c or Range of values of c which satisfy

$$-1 < \underbrace{1+3c < 1}$$

$-1 < 1+3c$

$-1 \quad -1$

$-2 < 3c$

$1+3c < 1-1$

-1

$\Downarrow$

$3c < 0$

$$\frac{-2}{3} < \frac{3c}{3}$$

$$\frac{3c}{3} < \frac{0}{3}$$

$$\boxed{-2/3 < c}$$

$$\boxed{c<0}$$

$$\boxed{-2/3 < c < 0}$$

$$x_{n+1} = x_n + c\,(\underbrace{x_n^3 - 1}_{f(x)})$$

<u>6/3.4 Page 107</u>: Convert $x^2-5=0$ to the fixed-point problem:

$$x = x + c\,(x^2-5) := g(x)$$

definition notation

with nonzero c.

Detemine possible values of c that ensure convergence of

$$x_{n+1} = x_n + c\,(\underbrace{x_n^2 - 5}_{f(x)})$$

to $\underline{\underline{\alpha = \sqrt{5}}}$.

$$x_{n+1} = g(x_n), \quad g(x_n) = x_n + c(x_n^2 - 5)$$

<u>use</u>: convergence of $x_{n+1} = g(x_n)$ is guaranteed

if $$-1 < g'(\alpha) < 1$$

$$g(x) = x + c\,(x^2-5)$$

control parameter chosen to guarantee convergence.

$$g'(x) = 1 + 2cx$$

convergence."

$g'(x) = 1 + -2cx$

$\alpha = \sqrt{5}$

$g'(\alpha) = 1 + 2\sqrt{5}c$ (Warning: $g'(x)$ deri. wrt x but replace "c" with α MISTAKE)

$$-1 < g'(\alpha) < 1$$

$$-1 < 1 + 2\sqrt{5}c < 1$$

$-1 < 1 + 2\sqrt{5}c$

$-1 \quad -1$

$\frac{-2}{2\sqrt{5}} < \frac{2\sqrt{5}c}{2\sqrt{5}}$

$-\frac{2}{2\sqrt{5}} < c$

$\boxed{-1/\sqrt{5} < c}$

$1 + 2\sqrt{5}c < 1$

$-1 \quad -1$

$\frac{2\sqrt{5}c}{2\sqrt{5}} < \frac{0}{2\sqrt{5}}$

$\boxed{c < 0}$

$\boxed{-1/\sqrt{5} < c < 0}$

check if the following iterative method converges to α

$x_{n+1} = 3/4\, x_n + 1/x_n^3 \qquad \alpha = \sqrt{2}$

provided x_0 is close to α.

$g(x) = 3/4\, x + 1/x^3$

check if $-1 < g'(\alpha) < 1$

$g(x) = \frac{3}{4}x + \frac{1}{x^3} \rightarrow g'(x) = \frac{3}{4} - \frac{3}{x^4}$

$g(x) = \frac{3}{4}x + \frac{1}{x^3}, \quad g'(x) = \frac{3}{4} - \frac{3}{x^4}$

$\alpha = \sqrt{2} \quad \alpha^2 = 2 \quad \alpha^4 = 2^2 = 4$

$g'(\alpha) = \frac{3}{4} - \frac{3}{4} = 0 \longrightarrow$ lives in interval -1 to 1

so $x_{n+1} = g(x_n)$ converges to $\alpha = \sqrt{2}$.

$x_{n+1} = g(x_n) \quad \{x_{n+1}\} \longrightarrow \alpha$ if x_0 is close to α

and $-1 < g'(\alpha) < 1$

$$\boxed{g(x) = \frac{3}{4}x + \frac{1}{x^3}}$$ No "c"

x_0 guess

$$x_1 = \frac{3}{4}x_0 + \frac{1}{x_0^3}$$

$$g'(\alpha) = 3/4 - 3/\alpha^4$$ No c

$$-1 < g'(\alpha) < 1 \ ?$$

$$g'(\alpha) = 3/4 - 3/(\sqrt{2})^4 = 3/4 - 3/4$$
$$= 0 < 1$$

$-1 <$

11/sectn 3.4 The following

$$x_{n+1} = 2 - (1+c)x_n + cx_n^3$$

will converge to α for some values of c.

Please determine the values of c for which $x_{n+1} = 2 - (1+c)x_n + cx_n^3$ converges

Please determine the values of c … $x_{n+1} = \underbrace{2-(1+c)x_n + c x_n^3}_{g(x_n)}$ converges

Pick c such that:

$$-1 < g'(\alpha) < 1, \quad \alpha = 1$$

$$-1 < 2c-1 < 1$$

$$0 < c < 1$$

$$g(x) = 2-(1+c)x + cx^3 \rightarrow 2 - x - cx + cx^3$$

$$g'(x) = -(1+c) + 3cx^2 \qquad g'(x) = 0 - 1 - c + 3cx^2$$

$x \to \alpha = 1$ $\qquad g'(\alpha) = -1 - c + 3c$

$$g'(\alpha) = -1-c+3c = 2c-1$$

$$-1 < \underbrace{2c-1 < 1}_{2c < 2}$$

$$\boxed{c < 1}$$

$$-1 < 2c-1 \quad +1 \quad +1$$

$$0 < 2c$$

$$0 < c$$

$$\boxed{0 < c < 1}$$

Rate of Convergence of $x_{n+1} = \underline{g(x_n)}$

Rate of Convergence of $x_{n+1} = g(x_n)$

If $-1 < g'(\alpha) < 1$ then $x_{n+1} = g(x_n)$ converges.

If $g'(\alpha) \neq 0$ then convergence is linear.

$$|\alpha - x_{n+1}| < k\,|\alpha - x_n| \quad \text{where } 0 < k < 1$$

If $g'(\alpha) = 0$ & $g''(\alpha) \neq 0$ then convergence is quadratic.

$$|\alpha - x_{n+1}| < k\,|\alpha - x_n|^2, \quad k > 0$$

If $g'(\alpha) = 0$, $g''(\alpha) = 0$ and $g'''(\alpha) \neq 0$ then convergence is cubic

$$|\alpha - x_{n+1}| < k\,|\alpha - x_n|^3, \quad k > 0$$

<u>Example:</u> $x_{n+1} = \frac{3}{4}x_n + \frac{1}{x_n^3}$ $\quad n \geq 0, \ \alpha = \sqrt{2}$

$$g(x) = \frac{3}{4}x + \frac{1}{x^3}$$

$$g'(x) = \frac{3}{4} - \frac{3}{x^4} \qquad g'(\alpha) = \frac{3}{4} - \frac{3}{4} = 0$$

$g'(\alpha) = 0 \longrightarrow$ check $g''(\alpha) = 0$ or not

$$g'(x) = \frac{3}{4} - \frac{3}{x^4}$$

$$g''(x) = 0 - 3 \times (-4/ \ldots)$$

$$g''(x) = 0 - 3 * (-4/x^5)$$

$$g''(\alpha) = \frac{12}{\alpha^5}, \quad \alpha = \sqrt{2}$$

$$\neq 0 \quad \Rightarrow \quad x_{n+1} = \frac{3}{4}x_n + \frac{1}{x_n^3}$$

Converges quadratically.

11/3.4 $\qquad x_{n+1} = 2 - (1+c)x_n + c\,x_n^3$

$$g(x) = 2 - (1+c)x + cx^3, \quad \alpha = 1$$

$$g'(1) = -1 - c + 3c$$

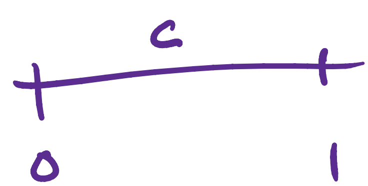
$$g'(\alpha) = -1 + 2c$$

convergence guaranteed if

$$-1 < -1 + 2c < 1$$

$$\boxed{0 < c < 1}$$

can we find c values for which convergence is quadratic?



$$g'(x) = -1 - c + 3cx^2$$

$$\boxed{g'(\alpha) = 0 \ \& \ g''(\alpha) \neq 0}$$ QUADRATIC CONVERGENCE

$$g'(1) = -1 - c + 3c = 2c - 1$$

$g'(1) = -1 - c + 3c = 2c - 1$

$g'(1) = 0 \longrightarrow \boxed{2c-1} = 0 \longrightarrow c = 1/2$

$g''(1) \neq 0$?

$g'(x) = -1 - c + 3cx^2$

$g''(x) = 0 + 6cx \qquad g''(1) = 6c \neq 0$