Math 4329

Tuesday, March 3, 2020 4:32 PM

Purpose of "Algebraic Questions" on homework.

#3 $f(x) = x^2 - a$

a) $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} \longrightarrow x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$

b) Derive absolute error $\sqrt{a} - x_{n+1} = -\frac{1}{2x_n}(\sqrt{a} - x_n)^2$

Relative error $Rel(x_{n+1}) = -\frac{\sqrt{a}}{2x_n}(Rel(x_n))^2$

c) $x_0 \approx \sqrt{a}$ (Last Lect: $x_0 = 1.7$ $a = 3$ root $= \sqrt{3} \approx 1.732$)

$Rel(x_0) = 0.1$ $Rel(x_{n+1}) \approx -\frac{1}{2} Rel(x_n)^2$

Compute Relative(x_1), Rel.(x_2), Rel(x_3), Rel(x_4)

$\downarrow n=0$

$Rel(x_1) = -\frac{1}{2}(0.1)^2$

⑧ $x = \frac{1}{3}$?

Goal: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ simplify form.

$f(x) = x^2 - a$ then,

$x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$

$f(x) = x^2 - 5$

2 $x_n)$

$f(x) = x^2 - 5$

$$\boxed{x_{n+1} = \frac{1}{2}\left(x_n + \frac{5}{x_n}\right)}$$ Relevant for today!

W.S 05: $f(x) = -x^4 + 3x^2 + 2$

$x_0 = 1$

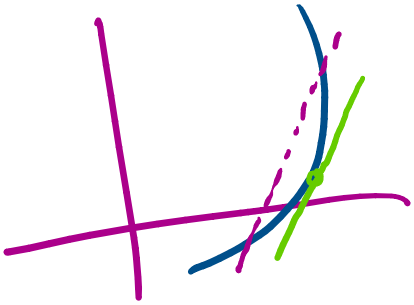
$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 1 - \frac{(-1+3+2)}{-4+6}$$

$$= 1 - (4/2) = -1$$

$$x_2 = -1 - \frac{\overset{4}{(-1+3+2)}}{\underset{4-6}{-4(-1)+6(-1)}} = -1 - (-4/2) = -1+2 = 1$$

$$\boxed{f'(x) = -4x^3 + 6x \quad , x = 0 \quad f'(0) = 0 \quad (x-1)^2}$$

Secant method $$x_{n+1} = x_n - \frac{f(x_n)}{\frac{f(x_n)-f(x_{n-1})}{x_n - x_{n-1}}}$$



$$\boxed{x_{n+1} = x_n - f(x_n)\left(\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}\right)}$$

Initial x_0, x_1 → TWO STEP METHOD

then $$x_2 = x_1 - f(x_1)\left(\frac{x_1 - x_0}{f(x_1) - f(x_0)}\right)$$

Simplifying the formula ↑

Secant Method: Simplifying the formula

Problem → complicated formula needs simplification!

Apply secant method to solve $x^3 - 1 = 0$.

Simplify the formula using:

$$a^3 - b^3 = (a-b)(a^2 + b^2 + ab)$$

Simplify $$x_{n+1} = x_n - f(x_n)\left(\frac{x_n - x_{n-1}}{f(x_n) - f(x_{n-1})}\right)$$

[Similar task to:
3(a)/3.2 : $x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$]

Replace $f(x_n) = x_n^3 - 1$.

$$x_{n+1} = x_n - (x_n^3 - 1)\frac{(x_n - x_{n-1})}{\left((x_n^3 - 1) - (x_{n-1}^3 - 1)\right)}$$

$$= x_n - (x_n^3 - 1)\frac{(x_n - x_{n-1})}{(\underbrace{x_n^3 - 1 - x_{n-1}^3 + 1}_{x_n^3 - x_{n-1}^3})}$$

$$= x_n - (x_n^3 - 1)\frac{(x_n - x_{n-1})}{(\underset{a}{x_n^3} - \underset{b}{x_{n-1}^3})}$$

use $a^3 - b^3 = (a-b)(a^2 + b^2 + ab)$

use $a^3-b^3 = (a-b)(a^2+b^2+ab)$

$$= x_n - \frac{(x_n^3-1)\,\cancel{(x_n - x_{n-1})}}{\cancel{(x_n-x_{n-1})}\,(x_n^2+x_{n-1}^2+x_nx_{n-1})}$$

$$= x_n - \frac{(x_n^3-1)}{\boxed{x_n^2+x_{n-1}^2+x_nx_{n-1}}}$$

$$= \frac{x_n(x_n^2+x_{n-1}^2+x_nx_{n-1}) - (x_n^3-1)}{(x_n^2+x_{n-1}^2+x_nx_{n-1})}$$

$$= \frac{\cancel{x_n^3} + x_nx_{n-1}^2 + x_n^2x_{n-1} - \cancel{x_n^3} + 1}{x_n^2+x_{n-1}^2+x_nx_{n-1}}$$

careful!

$$x_{n+1} = \frac{x_nx_{n-1}^2 + x_n^2x_{n-1} + 1}{x_n^2+x_{n-1}^2+x_nx_{n-1}}$$

compare with

$$x_{n+1} = x_n - \frac{(x_n^3-1)(x_n-x_{n-1})}{(\underset{-1}{x_n^3} - \underset{+1}{x_{n-1}^3})}$$

Design of methods

Design of Iterative Methods

3.4

$x^2 - 5 = 0$ $\boxed{\alpha = \sqrt{5}}$

Solve $f(x) = 0$ it is the same as

solving $x + f(x) = x$

$g(x) = x$, $g(x) = x + f(x)$

$g(x) = x \longrightarrow$ x is a fixed point

$$f(x) = 0 \longrightarrow x = g(x)$$

$$f(x) = 0 \longleftarrow x = g(x)$$

$x^2 - 5 = 0$ New formulas of form $x_{n+1} = g(x_n)$

example: $g(x) = x - f(x)/f'(x)$ then

$x_{n+1} = g(x_n)$ is Newton's method formula.

I1: Goal $x_{n+1} = g_1(x_n) = x_n + 5 - x_n^2$

$$x^2 - 5 = 0 \longrightarrow 5 - x^2 = 0$$

$$x + 5 - x^2 = x + 0$$

$$x_n + 5 - x_n^2 = x_{n+1}$$

$$\boxed{x_{n+1} = x_n + 5 - x_n^2.}$$

$$\boxed{x_{n+1} = x_n + 5 - x_n^2 .}$$

I2: Goal $\boxed{x_{n+1} = g_2(x_n) = 5/x_n}$

$x^2-5=0 \longrightarrow x^2-5+5=0+5$

$$\frac{x^2}{x} = \frac{5}{x}$$

$$\boxed{x_{n+1} = \frac{5}{x_n}}$$

I3: Goal $x_{n+1} = g_3(x_n) = x_n - x_n^2/5 + 1$

$x^2-5=0 \longrightarrow \frac{x^2-5}{5} = 0$

$$-1 * \frac{x^2}{5} - 1 = 0$$

$$x - \frac{x^2}{5} + 1 = 0 + x$$

$$x_n - \frac{x_n^2}{5} + 1 = x_{n+1}$$

$$\boxed{x_{n+1} = x_n - \frac{x_n^2}{5} + 1}$$

I4: Goal $\boxed{x_{n+1} = g_4(x_n) = \frac{1}{2x_n}(x_n^2+5)}$ Newton's Method

$x^2-5=0 \longrightarrow 2x^2 - x^2 - 5 = 0 + x^2 + 5$

$+ x^2 + 5$

$$2x^2 = x^2 + 5$$

$x - 5 = 0$

$+ x^2+5 \quad 2x^2 = x^2+5$

$$\frac{2x^2}{2x} = \frac{x^2+5}{2x}$$

$$x_{n+1} = \frac{1}{2x_n}(x_n^2+5)$$

How to predict convergence of

I1: $x_{n+1} = g_1(x_n)$

where $g_1(x) = 5+x-x^2$

If $-1 < g_1'(\alpha) < 1$

then $x_{n+1} = g_1(x_n)$ generates a sequence $\{x_n\}_{n\geq 0}$ converges to α.

$g_1'(x) = 0+1-2x$

$g_1'(\sqrt{5}) = 1-2\sqrt{5} < -1$

NO CONVERGENCE!

I3: $x_{n+1} = g_3(x_n)$

$g_3(x) = 1+x-x^2/5$

$$g_3'(x) = 0+1-\frac{2x}{5}$$

$$g_3'(\alpha) = 1-\frac{2\sqrt{5}}{5}$$

$= 1-2/\sqrt{5}$

$\approx 0.106 < 1$

$\Rightarrow$ $x_{n+1} = g_3(x_n)$

$\{x_n\}$ generated

converges to $\alpha = \sqrt{5}$

I2: $g_2(x) = 5/x$

$g_2'(x) = -5/x^2$

$$\boxed{-1 < g_2'(\alpha) < 1}$$

$\alpha = \sqrt{5}$

$g_2'(x) = -5/x^2$

$\alpha = \sqrt{5}$

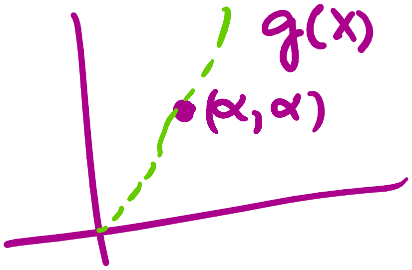
$g_2'(\alpha) = -5/5 = -1$

1

-1

INCONCLUSIVE

$$\boxed{|g'(\alpha)| < 1}$$



$|g'(\alpha)| < 1$